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An algebraic scheme associated with the non-commutative KP hierarchy and some of its extensions

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Abstract

A well-known ansatz ('trace method') for soliton solutions turns the equations of the (non-commutative) KP hierarchy, and those of certain extensions, into families of algebraic sum identities. We develop an algebraic formalism, in particular involving a (mixable) shuffle product, to explore their structure. More precisely, we show that the equations of the non-commutative KP hierarchy and its extension (xncKP) in the case of a Moyal-deformed product, as derived in previous work, correspond to identities in this algebra. Furthermore, the Moyal product is replaced by a more general associative product. This leads to a new even more general extension of the non-commutative KP hierarchy. Relations with Rota–Baxter algebras are established.

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1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero and $(\mathcal{R}_0, *)$ the $\mathbb{K}$-algebra of differential polynomials in (matrices of) functions $\{u_{n+1} | n \in \mathbb{N}\}$ of variables $t_n, n \in \mathbb{N}$, with an associative (and non-commutative) product $*$ for which the operators of partial differentiation with respect to $t_n, n \in \mathbb{N}$ are derivations[3]. A formal pseudo-differential operator (ΨDO) in the following means a formal series in the operator[4] ∂ of partial differentiation with respect to $x := t_1$ and

[3] Here and in the following $\mathbb{N}$ denotes the natural numbers *not* including zero.

[4] An expression like ∂X has to be understood as a product of operators, whereas $\partial_x X$ will be used for the partial derivative of X with respect to x, also denoted as X_x.

its formal inverse ∂^{-1} with coefficients in $(\mathcal{R}_0, *)$. With elements $f \in \mathcal{R}_0$, ∂^{-1} satisfies the relation

$$\partial^{-1} f = f\partial^{-1} - f_x\partial^{-2} + f_{xx}\partial^{-3} - \cdots. \tag{1.1}$$

We will use $(\,)_{\geqslant 0}$ and $(\,)_{<0}$, respectively, to denote the projection to that part of a ΨDO which only contains non-negative, respectively negative, powers of ∂. Let $\mathcal{R}$ be the ring of ΨDOs generated by

$$L = \partial + \sum_{n\geqslant 1} u_{n+1}\partial^{-n} \tag{1.2}$$

using the product $*$, the projections and the defining relation for ∂^{-1} as the inverse of $\partial = L_{\geqslant 0}$. This is also a $\mathbb{K}$-algebra. In the Sato framework, the non-commutative KP hierarchy (ncKP) is defined by[5]

$$L_{t_n} := \partial_{t_n} L = [(L^n)_{\geqslant 0}, L] = -[(L^n)_{<0}, L] \qquad n = 1, 2, \ldots \tag{1.3}$$

(see [1–7], for example). Introducing a potential ϕ via

$$u_2 = \phi_x \tag{1.4}$$

one finds the following expressions for the commuting flows of the ncKP hierarchy[6]:

$$\phi_{t_n} = \operatorname{res}(L^n) \qquad n = 1, 2, \ldots. \tag{1.5}$$

Let us now recall a method[7] [8] to obtain soliton solutions of the (potential) *ncKP equation*

$$\left(4\phi_{t_3} - \phi_{xxx} - 6\phi_x * \phi_x\right)_x = 6[\phi_y, \phi_x] + 3\phi_{yy} \tag{1.6}$$

where $y := t_2$. This is the first non-trivial member of the ncKP hierarchy. Inserting the formal series

$$\phi = \sum_{N=1}^{\infty} \epsilon^N \phi^{(N)} \tag{1.7}$$

in a parameter ϵ, transforms it into the system of equations

$$4\phi^{(N)}_{t_3 x} - \phi^{(N)}_{xxxx} - 3\phi^{(N)}_{yy} = 6\sum_{k=1}^{N-1}\left((\phi^{(k)} * \phi^{(N-k)})_x + \left[\phi^{(k)}_y, \phi^{(N-k)}_x\right]\right) \tag{1.8}$$

which is solved by

$$\phi^{(k)} = \sum_{i_1,\ldots,i_k=1}^{M} \frac{\phi_{i_1} * \phi_{i_2} * \cdots * \phi_{i_k}}{(q_{i_1} - p_{i_2})(q_{i_2} - p_{i_3})\cdots(q_{i_{k-1}} - p_{i_k})} \qquad k = 1, \ldots, N \tag{1.9}$$

with

$$\phi_k = c_k\, \mathrm{e}^{\xi(t,p_k)} * \mathrm{e}^{-\xi(t,q_k)} \tag{1.10}$$

where $M \in \mathbb{N}$, $\xi(t, p_k) = \sum_{r\geqslant 1} t_r p_k^r$ (see also [2, 6, 9]).[8] Here c_k, p_k, q_k are constants such that c_k and the denominators in (1.9) are different from zero. Inserting (1.9) with (1.10) in (1.8) first leads to a sum which runs over all lists $(i_1, \ldots, i_N)$ where $i_k \in \{1, \ldots, M\}$. But it

[5] Here L^n stands for the n-fold product $L * \cdots * L$, and $[\,,\,]$ is the commutator in the ring $(\mathcal{R}, *)$.

[6] To be precise, here we need to supply $\mathcal{R}$ with the operation of x-integration. The residue of a ΨDO is the coefficient of its ∂^{-1} term.

[7] For a different method in the non-commutative setting, see [1], for example.

[8] M is the soliton number. For $M = 1$ we can use the geometric series formula (in the domain of convergence of the series) to obtain $\phi = \sum_{N=1}^{\infty}(\epsilon\phi_1/(q_1 - p_1))^N = (1 - \epsilon\phi_1/(q_1 - p_1))^{-1} - 1$ from which one recovers a well-known expression for the 1-soliton solution of the KP equation.

actually results in *separate* sum identities (of the same kind), involving the constants p_k, q_k. It is therefore sufficient to consider only the terms corresponding to one definite representative list, say those proportional to $\phi_1 * \cdots * \phi_N$ (where some of the ϕ_k may be equal)[9]. For example, the corresponding contribution of the expression

$$\phi_{t_r}^{(N)} = \sum_{i_1,\ldots,i_N=1}^{M} \sum_{k=1}^{N} \left(p_{i_k}^r - q_{i_k}^r\right) \frac{\phi_{i_1} * \cdots * \phi_{i_N}}{(q_{i_1} - p_{i_2}) \cdots (q_{i_{N-1}} - p_{i_N})} \tag{1.11}$$

is $T_r \phi_1 * \cdots * \phi_N \big/ \prod_{k=1}^{N-1} (q_k - p_{k+1})$ where

$$T_r := \sum_{k=1}^{N} \left(p_k^r - q_k^r\right). \tag{1.12}$$

The Nth order part (1.8) of the ncKP equation is then mapped to the following algebraic equation[10]

$$4T_1T_3 - {T_1}^4 - 3{T_2}^2 = 6T_1(T_1 \times T_1) - 6(T_1 \times T_2 - T_2 \times T_1) \tag{1.13}$$

where

$$T_r \times T_s := \sum_{1 \leqslant i \leqslant j < k \leqslant N} \left(p_i^r - q_i^r\right) q_j \left(p_k^s - q_k^s\right) - \sum_{1 \leqslant i < j \leqslant k \leqslant N} \left(p_i^r - q_i^r\right) p_j \left(p_k^s - q_k^s\right). \tag{1.14}$$

Equation (1.8) is solved by (1.9) if (1.13) is an identity, which indeed turns out to be the case on closer inspection. Note that this identity not only holds for arbitrary values of the p_k, q_k, but also for arbitrary $N \in \mathbb{N}$.[11] Inspection of the identity (1.13) suggests a way to obtain such identities directly from ncKP equations. The basic rules are[12]

$$\phi_{t_{m_1} \ldots t_{m_k}} \mapsto T_{m_1} \cdots T_{m_k} \qquad \phi_{t_r} * \phi_{t_s} \mapsto T_r \times T_s. \tag{1.15}$$

Now (1.13) immediately follows from (1.6).

Taking (1.7) with (1.9) as an *ansatz* to obtain solutions of a partial differential equation involving the product $*$ and partial derivatives of a field ϕ with respect to the variables t_n turns it into an algebraic equation. If this is an identity for all N, the respective equation has KP-type soliton solutions[13]. Does the ncKP hierarchy exhaust the possibilities of such equations?

In particular, we will be interested in the case where the product $*$ depends on parameters. An example is given by the (Groenewold–) Moyal product [10–13]

$$f * g := \mathbf{m} \circ \mathrm{e}^{P/2}(f \otimes g) \qquad P := \sum_{m,n=1}^{\infty} \theta_{mn} \partial_{t_m} \otimes \partial_{t_n} \tag{1.16}$$

[9] For $N > M$ some of the ϕ_k necessarily have to be equal, but for $N \leqslant M$ there are lists for which all factors ϕ_k are potentially independent. We should consider such lists as 'representative'. We thus assume that the soliton number M can be chosen arbitrarily large. The number M then does not enter the subsequent considerations any more.

[10] We should stress again that each summand in (1.9) leads separately to this identity. The non-commutative terms on the right-hand side of (1.13) still remain present if we let the product $*$ become commutative, i.e., in the case of the 'commutative' KP hierarchy. In that case, they disappear, however, via the summation in (1.9).

[11] Such identities typically decompose into several identities since terms such as $\sum_{1 \leqslant i \leqslant j < k \leqslant N} p_i p_j p_k$ and $\sum_{1 \leqslant j < k \leqslant N} p_j p_k^2$, for example, are obviously linearly independent. Accordingly, one can introduce a notion of *length*, so that expressions decompose into linearly independent parts of fixed length. In the step towards the algebra developed in section 2 we abstracted the above sums to products $P \prec P \prec P$, respectively $P \prec P \bullet P$, from which they are recovered via a representation Σ_N (see section 3). The grading given by $\prec$ takes care of the length.

[12] In the usual formulation of the ncKP hierarchy, ϕ without derivatives acting on it does not appear, so we need not say to what a bare ϕ should correspond.

[13] Of course, one may think of modifications of (1.9) in the search for other equations admitting a soliton structure.

where $\mathbf{m}(f \otimes g) = fg$ for functions f, g, and $\theta_{nm} = -\theta_{mn}$ are parameters. Then there is another basic rule, namely

$$\partial_{\theta_{rs}} \mapsto \Theta_{rs} := \frac{1}{2} \sum_{1\leqslant j<k\leqslant N} \left[\left(p_j^r - q_j^r\right)\left(p_k^s - q_k^s\right) - \left(p_j^s - q_j^s\right)\left(p_k^r - q_k^r\right)\right] \\ - \frac{1}{2} \sum_{k=1}^{N} \left(p_k^r q_k^s - p_k^s q_k^r\right). \tag{1.17}$$

According to our correspondence rules, we have, for example,

$$\phi_{\theta_{rs} t_k} * \phi_{t_l} * \phi_{t_m} \mapsto (T_k \Theta_{rs}) \times T_l \times T_m \tag{1.18}$$

with a (rather obvious) generalization of (1.14) which defines an associative product (of sums of powers of $p_1, \ldots, p_N, q_1, \ldots, q_N$). The first equation of the extension (in the sense of [5–7]) of the ncKP hierarchy (with Moyal $*$-product), called the *xncKP hierarchy*, is

$$\phi_{\theta_{1,2}} = \tfrac{1}{6}\left(\phi_{t_3} - \phi_{xxx}\right) - \phi_x * \phi_x. \tag{1.19}$$

This is mapped to

$$\Theta_{1,2} = \tfrac{1}{6}\left(T_3 - {T_1}^3\right) - T_1 \times T_1 \tag{1.20}$$

which indeed also turns out to be an identity.

Hence, taking (1.7) with (1.9) as an ansatz to obtain solutions of a (in this case non-local) partial differential equation involving the Moyal product and partial derivatives of a field ϕ with respect to the variables t_r and θ_{mn} converts it into an algebraic equation. If this is an identity for all N, the respective equation has KP-type soliton solutions. The equations of the xncKP hierarchy provide us with corresponding examples.

The mapping of (x)ncKP equations to algebraic identities described above can actually be reversed. From (1.13), respectively (1.20), we easily reconstruct the partial differential equations (1.6), respectively (1.19). It should be clear that, in order to do this, the sum calculus is not essential, but rather a certain algebraic abstraction. This motivates us to develop an algebraic scheme which allows us to prove and to find identities of the kind we met above. The way in which we expressed the identities (1.13) and (1.20) already suggests some main ingredients of such a scheme. A deeper analysis led us to the algebra which we introduce in section 2. A correspondence between identities holding in the abstract algebra and the equations of the ncKP hierarchy and certain extensions is indeed established in this work. In this context one should keep in mind that characteristic properties of the KP hierarchy are indeed purely algebraic. In particular, this concerns the basic property of commutativity of the flows. Writing (1.3) in the form

$$\partial_{t_n} L = \delta_n L \qquad \delta_n L := [(L^n)_{\geqslant 0}, L] \tag{1.21}$$

and extending δ_n to $\mathcal{R}$ according to the derivation rule (together with $\delta_m X_{\geqslant 0} := (\delta_m X)_{\geqslant 0}$ for $X \in \mathcal{R}$), the commutativity of the flows becomes equivalent to

$$[\delta_m, \delta_n] L = 0 \tag{1.22}$$

which is a purely algebraic identity in the ring $\mathcal{R}$ (and in particular makes no reference to the variables $t_n, n > 1$). Associated with the extension of the Moyal-deformed KP hierarchy are 'generalized derivations' which also commute as a consequence of algebraic identities. We will meet even more generalized derivations in section 6. They also define extensions of the KP hierarchy with a deformed product (see section 8).

The treatment of the xncKP hierarchy in [5–7] heavily relies on the fact that the underlying algebra $\mathcal{R}$ of ΨDOs admits the decomposition $\mathcal{R} = \mathcal{R}_{\geqslant 0} \oplus \mathcal{R}_{<0}$ into subalgebras, whereas in the treatment of the ncKP hierarchy it is sufficient to have a corresponding decomposition of Lie algebras (as common in integrable systems theory)[14]. Such an algebra decomposition is equivalent to the existence of an idempotent Rota–Baxter operator R [14–17] on the algebra (see also appendix A). A few years ago, it was shown that the choice of a renormalization scheme in perturbative quantum field theory corresponds to the choice of a Rota–Baxter operator [18–21]. In [22, 23] it has been pointed out that this setting resembles the loop algebra framework of integrable systems. The antisymmetric part of the bilinear Rota–Baxter relation (of weight 1) is the famous *classical Yang–Baxter relation*, which plays an important role in integrable system theory [24–26]. It should not come as a surprise that various Rota–Baxter relations also appear in the present work.

Section 2 introduces the algebra $\mathcal{A}$ which plays a basic role in this work. Section 3 then provides a realization in terms of partial sum calculus. Some other realizations of the algebra $\mathcal{A}$ are briefly described in appendix B. Section 4 treats the case of the subalgebra $\mathcal{A}(P)$ of $\mathcal{A}$ generated by a single element P. This plays a central role in the subsequent sections. Section 5 addresses the case of a subalgebra of $\mathcal{A}$ generated by two commuting elements and an embedding of $\mathcal{A}(P)$. Although this section is important in order to make contact with the aforementioned algebraic sum identities, it may be skipped on first reading. Sections 6 and 7 relate the algebraic framework with the ncKP hierarchy and (in the case where $*$ is the Moyal product) its xncKP extension. A more general extension, corresponding to a more general $*$-product (see appendix C), is studied in section 8. Appendix D sketches a certain generalization of the algebraic framework which, in particular, allows us to introduce an algebraic counterpart of a Baker–Akhiezer function (formal eigenfunction of a Lax operator like L). Section 9 contains some conclusions and further remarks.

2. The basic algebraic structure

Let $\mathcal{A} = \bigoplus_{r\geqslant 1} \mathcal{A}^r$ be a graded linear space over a field $\mathbb{K}$ of characteristic zero, which becomes an associative algebra with respect to two products $\prec$ and $\bullet$, which are bilinear maps $\mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s}$ and $\mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s-1}$, respectively[15]. Furthermore, we require that the two products satisfy the mutual associativity conditions

$$(\alpha \prec \beta) \bullet \gamma = \alpha \prec (\beta \bullet \gamma) \qquad (\alpha \bullet \beta) \prec \gamma = \alpha \bullet (\beta \prec \gamma) \tag{2.1}$$

for all $\alpha, \beta, \gamma \in \mathcal{A}$. It is convenient to introduce the notation

$$\alpha \succ \beta := \alpha \prec \beta + \alpha \bullet \beta \tag{2.2}$$

for the combined product which is clearly also associative. This new product induces a different grading of the algebra: $\mathcal{A} = \bigoplus_{r\geqslant 1} \mathcal{A}_r$, where $\mathcal{A}_1 = \mathcal{A}^1$ and $\mathcal{A}_r \succ \mathcal{A}_s \subseteq \mathcal{A}_{r+s}$. We also have $\mathcal{A}_r \bullet \mathcal{A}_s \subseteq \mathcal{A}_{r+s-1}$ and $\mathcal{A}_r \prec \mathcal{A}_s \subseteq \mathcal{A}_{r+s-1} \bigoplus \mathcal{A}_{r+s}$.

Let Shuff(m, n) denote the set of (m, n)-*shuffles*, i.e.

$$\mathrm{Shuff}(m, n) := \{\sigma \in \mathcal{S}_{m+n} | \sigma^{-1}(1) < \cdots < \sigma^{-1}(m), \sigma^{-1}(m+1) < \cdots < \sigma^{-1}(m+n)\} \tag{2.3}$$

where $\mathcal{S}_n$ is the symmetric group acting on n letters. For example,

$$\mathrm{Shuff}(1, n) = \{\{1, 2, \ldots, n+1\}, \{2, 1, 3, \ldots, n+1\}, \ldots, \{2, 3, \ldots, n+1, 1\}\}$$

$$\mathrm{Shuff}(2, 2) = \{\{1, 2, 3, 4\}, \{1, 3, 2, 4\}, \{1, 3, 4, 2\}, \{3, 1, 2, 4\}, \{3, 1, 4, 2\}, \{3, 4, 1, 2\}\}$$

[14] Also $\mathcal{R} = \mathcal{R}_{\geqslant 1} \oplus \mathcal{R}_{<1}$ is a decomposition of the algebra of ΨDOs into subalgebras. This underlies the (extended) modified KP hierarchy (see [2, 7], for example).

[15] The grading basically accounts for the notion of 'length' mentioned in a previous footnote.

where a permutation σ is described by the ordered set $\{\sigma(1), \ldots, \sigma(m+n)\}$. Taking a deck of m cards and another one of n cards, $\mathrm{Shuff}(m,n)$ describes all possible shuffles of the two decks. It has $(m+n)!/(m!n!)$ elements. Clearly, $\mathrm{Shuff}(m,n) = \mathrm{Shuff}(n,m)$.

We define the *main product* $\circ$ in $\mathcal{A}$ by

$$\begin{aligned}(A_1 \Yup_1 \ldots \Yup_{m-1} A_m) \circ (A_{m+1} \Yup_{m+1} \ldots \Yup_{m+n-1} A_{m+n}) \\ := \sum_{\sigma \in \mathrm{Shuff}(m,n)} A_{\sigma(1)} \Yup'_{\sigma(1)} \ldots \Yup'_{\sigma(m+n-1)} A_{\sigma(m+n)}\end{aligned} \tag{2.4}$$

for $A_1, \ldots, A_{m+n} \in \mathcal{A}^1$. Each $\Yup_i$, $1 \leqslant i \leqslant m+n-1$, stands for one of the choices $\prec$ or $\succ$, and

$$\Yup'_{\sigma(i)} := \begin{cases} \succ & \text{if} \quad \sigma(i) \leqslant m < \sigma(i+1) \\ \prec & \text{if} \quad \sigma(i+1) \leqslant m < \sigma(i) \\ \Yup_i & \text{otherwise.} \end{cases} \tag{2.5}$$

This defines another associative product in $\mathcal{A}$. It is a *mixable shuffle product* [27, 28] with respect to the product pair $(\prec, \bullet)$, respectively $(\succ, \bullet)$. In particular, we find

$$\begin{aligned}(A_1 \Yup_1 A_2) \circ (A_3 \Yup_3 A_4) &= \sum_{\sigma \in \mathrm{Shuff}(2,2)} A_{\sigma(1)} \Yup'_{\sigma(1)} \ldots \Yup'_{\sigma(3)} A_{\sigma(4)} \\ &= A_1 \Yup_1 A_2 \succ A_3 \Yup_3 A_4 + A_1 \succ A_3 \prec A_2 \succ A_4 + A_1 \succ A_3 \Yup_3 A_4 \prec A_2 \\ &\quad + A_3 \prec A_1 \Yup_1 A_2 \succ A_4 + A_3 \prec A_1 \succ A_4 \prec A_2 + A_3 \Yup_3 A_4 \prec A_1 \Yup_1 A_2.\end{aligned} \tag{2.6}$$

Furthermore,

$$A_1 \circ A_2 = \sum_{\sigma \in \mathrm{Shuff}(1,1)} A_{\sigma(1)} \Yup'_{\sigma(1)} A_{\sigma(2)} = A_1 \succ A_2 + A_2 \prec A_1 \tag{2.7}$$

and, more generally,

$$\begin{aligned}A_1 \circ (A_2 \Yup_2 A_3 \Yup_3 \ldots \Yup_n A_{n+1}) &= \sum_{\sigma \in \mathrm{Shuff}(1,n)} A_{\sigma(1)} \Yup'_{\sigma(1)} \ldots \Yup'_{\sigma(n)} A_{\sigma(n+1)} \\ &= A_1 \succ A_2 \Yup_2 \ldots \Yup_n A_{n+1} + A_2 \prec A_1 \succ A_3 \Yup_3 \ldots \Yup_n A_{n+1} \\ &\quad + A_2 \Yup_2 A_3 \prec A_1 \succ A_4 \Yup_4 \ldots \Yup_n A_{n+1} + \cdots \\ &\quad + A_2 \Yup_2 A_3 \Yup_3 \ldots \Yup_n A_{n+1} \prec A_1\end{aligned} \tag{2.8}$$

where we can substitute either $\prec$ or $\succ$ for $\Yup_2, \ldots, \Yup_n$. Let $\beta = B_1 \Yup_1 B_2 \Yup_2 \ldots \Yup_{n-1} B_n$ with $B_i \in \mathcal{A}^1$ and $\beta_{[r,s]} := B_r \Yup_r \ldots \Yup_{s-1} B_s$ for $r \leqslant s$. The last formula can then be written more concisely as

$$A \circ \beta = A \succ \beta + \sum_{r=1}^{n-1} \beta_{[1,r]} \prec A \succ \beta_{[r+1,n]} + \beta \prec A. \tag{2.9}$$

It is convenient to introduce the 'Sweedler notation' [29]

$$A \circ \beta = A \succ \beta + \sum \beta_{(1)} \prec A \succ \beta_{(2)} + \beta \prec A. \tag{2.10}$$

In a similar way, we obtain

$$\beta \circ A = \beta \succ A + \sum \beta_{(1)} \succ A \prec \beta_{(2)} + A \prec \beta. \tag{2.11}$$

Remark. If $(\mathcal{A}^1, \bullet)$ is unital with a *unit element* E, this extends to $\mathcal{A}$ such that $E \bullet \alpha = \alpha = \alpha \bullet E$. Note that no rules are specified to resolve expressions such as $E \prec \alpha$ or $\alpha \prec E$.

2.1. Some properties of the algebra $\mathcal{A}$

Lemma 2.1. *Let $A \in \mathcal{A}^1$ and $\alpha, \beta \in \mathcal{A}$. Then*

$$A \circ (\alpha \curlywedge \beta) = (A \circ \alpha) \curlywedge \beta + \alpha \curlywedge (A \circ \beta) - \alpha \curlywedge A \curlywedge \beta \tag{2.12}$$
$$(\alpha \curlywedge \beta) \circ A = (\alpha \circ A) \curlywedge \beta + \alpha \curlywedge (\beta \circ A) - \alpha \curlywedge A \curlywedge \beta \tag{2.13}$$
$$[A, \alpha \curlywedge \beta]_\circ = [A, \alpha]_\circ \curlywedge \beta + \alpha \curlywedge [A, \beta]_\circ \tag{2.14}$$

where $[\,,\,]_\circ$ denotes the commutator with respect to the product $\circ$.

Proof. Because of linearity, it is sufficient to consider the case where $\alpha \in \mathcal{A}^m$ and $\beta \in \mathcal{A}^n$ for $m, n \in \mathbb{N}$. Using (2.10), we find

$$\begin{aligned} A \circ (\alpha \curlywedge \beta) &= A \succ (\alpha \curlywedge \beta) + \sum (\alpha \curlywedge \beta)_{(1)} \prec A \succ (\alpha \curlywedge \beta)_{(2)} + (\alpha \curlywedge \beta) \prec A \\ &= (A \succ \alpha) \curlywedge \beta + \sum \alpha_{(1)} \prec A \succ \alpha_{(2)} \curlywedge \beta + \alpha \prec A \succ \beta \\ &\quad + \alpha \curlywedge \sum \beta_{(1)} \prec A \succ \beta_{(2)} + \alpha \curlywedge \beta \prec A \\ &= (A \circ \alpha) \curlywedge \beta + \alpha \curlywedge (A \circ \beta) + \alpha \prec A \succ \beta - \alpha \prec A \curlywedge \beta - \alpha \curlywedge A \succ \beta. \end{aligned}$$

For both choices $\prec$ and $\succ$ for $\curlywedge$ this yields the first identity of the lemma. The second is obtained in the same way using (2.11). The third identity is an immediate consequence of the first two. $\square$

In the following we will adopt the convention that the product $\circ$, which does not satisfy mutual associativity relations with the other products, takes precedence over the other products. This means that it has to be evaluated first in expressions also containing other products. For example,

$$\alpha \circ \alpha' \curlywedge \beta \circ \beta' := (\alpha \circ \alpha') \curlywedge (\beta \circ \beta'). \tag{2.15}$$

Lemma 2.2.

$$(\alpha \prec A) \circ \beta = \alpha \prec A \succ \beta + \sum \alpha \circ \beta_{(1)} \prec A \succ \beta_{(2)} + \alpha \circ \beta \prec A \tag{2.16}$$

$$(A \succ \alpha) \circ \beta = A \succ \alpha \circ \beta + \sum \beta_{(1)} \prec A \succ \alpha \circ \beta_{(2)} + \beta \prec A \succ \alpha \tag{2.17}$$

$$\beta \circ (A \prec \alpha) = \beta \succ A \prec \alpha + \sum \beta_{(1)} \succ A \prec \beta_{(2)} \circ \alpha + A \prec \beta \circ \alpha \tag{2.18}$$

$$\beta \circ (\alpha \succ A) = \beta \circ \alpha \succ A + \sum \beta_{(1)} \circ \alpha \succ A \prec \beta_{(2)} + \alpha \succ A \prec \beta. \tag{2.19}$$

Proof. According to the definition of the shuffle product $\circ$, which preserves the order of the components of each factor (and the product symbols between them), an expression like $(\alpha \prec A) \circ \beta$ means that we first have to shuffle A into β and afterwards shuffle α into the resulting expression, but now with the restriction that all components of α have to precede A. For example, in order to evaluate $(A_1 \prec A_2) \circ \beta$, we first compute

$$A_2 \circ \beta = A_2 \succ \beta + \sum \beta_{(1)} \prec A_2 \succ \beta_{(2)} + \beta \prec A_2.$$

Then we shuffle A_1 into this expression as follows:

$$(A_1 \prec A_2) \circ \beta = A_1 \prec A_2 \succ \beta + \sum (A_1 \circ \beta_{(1)}) \prec A_2 \succ \beta_{(2)} + (A_1 \circ \beta) \prec A_2.$$

This obviously generalizes to

$$(\alpha \prec A) \circ \beta = \alpha \prec A \succ \beta + \sum (\alpha \circ \beta_{(1)}) \prec A \succ \beta_{(2)} + (\alpha \circ \beta) \prec A$$

which is the first identity of this lemma. The others are obtained by similar considerations. □

The following identity characterizes the main product as a 'quasi-shuffle product' [30].

Proposition 2.1.

$$(A \prec \alpha) \circ (B \prec \beta) = A \prec \alpha \circ (B \prec \beta) + B \prec (A \prec \alpha) \circ \beta + (A \bullet B) \prec \alpha \circ \beta. \qquad (2.20)$$

Proof. Using (2.18) and (2.2), we obtain

$$(A \prec \alpha) \circ (B \prec \beta) = (A \prec \alpha) \succ B \prec \beta + (A \prec B + A \bullet B) \prec (\alpha \circ \beta)$$
$$+ A \prec \sum \alpha_{(1)} \succ B \prec (\alpha_{(2)} \circ \beta) + B \prec (A \prec \alpha) \circ \beta.$$

Formula (2.20) is now obtained by rewriting the first term on the right-hand side as follows, again with the help of (2.18),

$$A \prec \alpha \succ B \prec \beta = A \prec \Big(\alpha \circ (B \prec \beta) - \sum \alpha_{(1)} \succ B \prec \alpha_{(2)} \circ \beta - B \prec \alpha \circ \beta\Big). \qquad □$$

In a similar way, one can prove the following identity:

$$(A \succ \alpha) \circ (B \prec \beta) = A \succ \alpha \circ (B \prec \beta) + B \prec (A \succ \alpha) \circ \beta. \qquad (2.21)$$

Remark. With $A \in \mathcal{A}^1$ let us associate a map $R_A : \mathcal{A} \to \mathcal{A}$ via $R_A(\alpha) = A \prec \alpha$. Then (2.20) reads

$$R_A(\alpha) \circ R_B(\beta) = R_A(\alpha \circ R_B(\beta)) + R_B(R_A(\alpha) \circ \beta) + R_{A \bullet B}(\alpha \circ \beta). \qquad (2.22)$$

In particular, if $A \in \mathcal{A}^1$ satisfies $A \bullet A = -\mathrm{q}A$ with $\mathrm{q} \in \mathbb{K}$, then R_A defines a *Rota–Baxter operator* of weight q on $(\mathcal{A}, \circ)$ [14–17] (see also appendix A and [27, 31–33] for relations with shuffle algebras). Associated with a unit element E is thus a Rota–Baxter operator of weight -1. If $\mathrm{q} = 0$ and $\alpha = \sum_{n\geqslant 1} a_n A^{\prec n}$, $\beta = \sum_{n\geqslant 1} b_n A^{\prec n}$, we obtain $\alpha \circ \beta = \sum_{n\geqslant 1} c_n P^{\prec n}$ with $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, from which we recover the ring of *Hurwitz series* (divided power series) [34].

Theorem 2.1. *If* $[A, B]_\bullet := A \bullet B - B \bullet A$ *vanishes for all* $A, B \in \mathcal{A}^1$, *then* $(\mathcal{A}, \circ)$ *is a commutative algebra.*

Proof. First we note that $[A, B]_\circ = [A, B]_\bullet$. (2.10) and (2.11) lead to

$$[A, \beta]_\circ = [A, \beta]_\bullet + \sum (\beta_{(1)} \prec A \bullet \beta_{(2)} - \beta_{(1)} \bullet A \prec \beta_{(2)})$$
$$= \sum_{r=1}^{n} B_1 \curlywedge_1 \ldots \curlywedge_{r-1} [A, B_r]_\bullet \curlywedge_r \ldots \curlywedge_{n-1} B_n$$

for $\beta = B_1 \curlywedge_1 \ldots \curlywedge_{n-1} B_n$. This vanishes indeed as a consequence of our assumption. Furthermore, from (2.20) we obtain

$$[A \prec \alpha, B \prec \beta]_\circ = A \prec [\alpha, B \prec \beta]_\circ + B \prec [A \prec \alpha, \beta]_\circ$$
$$+ (A \bullet B) \prec \alpha \circ \beta - (B \bullet A) \prec \beta \circ \alpha.$$

Using our assumption, the last two terms combine to $(A \bullet B) \prec [\alpha, \beta]_\circ$. Hence this formula can be used to prove our general statement by induction on the grades of α and β. □

2.2. *Involutions interchanging $\prec$ and $\succ$*

There is a fundamental duality in the algebra $\mathcal{A}$ concerning the two operations $\prec$ and $\succ$. It is convenient to encode this duality in two involutions which exchange the two products and their gradings:

$$(\alpha \prec \beta)^{\psi} = \alpha^{\psi} \succ \beta^{\psi} \qquad (\alpha \prec \beta)^{\omega} = \beta^{\omega} \succ \alpha^{\omega} \tag{2.23}$$

where for all $A \in \mathcal{A}^1$ also $A^{\psi}, A^{\omega} \in \mathcal{A}^1$. Using the involution property $\gamma^{\psi\psi} = \gamma$, respectively $\gamma^{\omega\omega} = \gamma$, for all $\gamma \in \mathcal{A}$, this implies

$$(\alpha \bullet \beta)^{\psi} = -\alpha^{\psi} \bullet \beta^{\psi} \qquad (\alpha \bullet \beta)^{\omega} = -\beta^{\omega} \bullet \alpha^{\omega}. \tag{2.24}$$

As a consequence,

$$(\alpha \succ \beta)^{\psi} = \alpha^{\psi} \prec \beta^{\psi} \qquad (\alpha \succ \beta)^{\omega} = \beta^{\omega} \prec \alpha^{\omega}. \tag{2.25}$$

We still have the freedom to define the action of the two involutions on the generators of $\mathcal{A}$.

Proposition 2.2.

$$(\alpha \circ \beta)^{\psi} = \beta^{\psi} \circ \alpha^{\psi} \qquad (\alpha \circ \beta)^{\omega} = \alpha^{\omega} \circ \beta^{\omega}. \tag{2.26}$$

Proof. By induction with respect to the grade of α. For $\alpha \in \mathcal{A}^1$ the identities easily follow from (2.10) and (2.11). If the identities hold for $\alpha \in \mathcal{A}^n$, they also hold for $\alpha \in \mathcal{A}^{n+1}$ by use of the identities (2.16) and (2.19). □

Applying the above involutions to identities in $\mathcal{A}$ generates further identities. This often provides us with a quick way of proving required relations.

Proposition 2.3.

$$(A \succ \alpha) \circ (B \succ \beta) = A \succ \alpha \circ (B \succ \beta) + B \succ (A \succ \alpha) \circ \beta - (B \bullet A) \succ \alpha \circ \beta \tag{2.27}$$

$$(\alpha \prec A) \circ (\beta \prec B) = \alpha \circ (\beta \prec B) \prec A + (\alpha \prec A) \circ \beta \prec B + \alpha \circ \beta \prec (A \bullet B) \tag{2.28}$$

$$(\alpha \succ A) \circ (\beta \succ B) = \alpha \circ (\beta \succ B) \succ A + (\alpha \succ A) \circ \beta \succ B - \alpha \circ \beta \succ (B \bullet A). \tag{2.29}$$

Proof. (2.27) and (2.29) are obtained by applying $^{\psi}$, respectively $^{\omega}$, to (2.20). (2.28) in turn results from (2.27) by application of $^{\omega}$ (or from (2.29) via $^{\psi}$). □

2.3. *Associative products determined by elements of $\mathcal{A}^1$*

With each $A \in \mathcal{A}^1$ we associate two bilinear maps $\hat{\mathbf{A}}, \check{\mathbf{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ via

$$\hat{\mathbf{A}}(\alpha, \beta) := \alpha \hat{\mathbf{A}} \beta := \alpha \prec A \succ \beta \tag{2.30}$$

$$\check{\mathbf{A}}(\alpha, \beta) := \alpha \check{\mathbf{A}} \beta := \alpha \succ A \prec \beta. \tag{2.31}$$

The 'product notation' is justified since the expressions on the right-hand sides are combined associative with all products defined so far, with the exception of the main product, and thus also among themselves. In particular, $(\alpha \check{\mathbf{A}} \beta) \hat{\mathbf{B}} \gamma = \alpha \check{\mathbf{A}} (\beta \hat{\mathbf{B}} \gamma)$ so that we are allowed to drop the brackets.

Lemma 2.3.

$$(\alpha \hat{\mathbf{A}} \beta)^{\psi} = \alpha^{\psi} \check{\mathbf{A}}^{\psi} \beta^{\psi} \qquad (\alpha \hat{\mathbf{A}} \beta)^{\omega} = \beta^{\omega} \hat{\mathbf{A}}^{\omega} \alpha^{\omega}. \tag{2.32}$$

Proof. These are immediate consequences of definitions (2.30) and (2.31), and the properties of the involutions $^{\psi}$ and $^{\omega}$ (see section 2.2). With $B := A^{\psi}$, $\check{\mathbf{A}}^{\psi}$ means $\check{\mathbf{B}}$. □

Proposition 2.4. *The following derivation properties of* $\circ$*-multiplication by an element* $B \in \mathcal{A}^1$ *hold:*

$$B \circ (\alpha \check{\mathbf{A}} \beta) = (B \circ \alpha) \check{\mathbf{A}} \beta + \alpha \check{\mathbf{A}} (B \circ \beta) \tag{2.33}$$

$$(\alpha \hat{\mathbf{A}} \beta) \circ B = \alpha \hat{\mathbf{A}} (\beta \circ B) + (\alpha \circ B) \hat{\mathbf{A}} \beta. \tag{2.34}$$

Proof. This is easily verified with the help of (2.12) and (2.13). Also note that the two identities are mapped to each other by application of the involution $^{\psi}$ (with $A^{\psi} = A$ for all $A \in \mathcal{A}^1$) and use of lemma 2.3. □

The next result is a generalization of the previous proposition.

Proposition 2.5.

$$\gamma \circ (\alpha \check{\mathbf{A}} \beta) = (\gamma \circ \alpha) \check{\mathbf{A}} \beta + \sum (\gamma_{(1)} \circ \alpha) \check{\mathbf{A}} (\gamma_{(2)} \circ \beta) + \alpha \check{\mathbf{A}} (\gamma \circ \beta) \tag{2.35}$$

$$(\alpha \hat{\mathbf{A}} \beta) \circ \gamma = \alpha \hat{\mathbf{A}} (\beta \circ \gamma) + \sum (\alpha \circ \gamma_{(1)}) \hat{\mathbf{A}} (\beta \circ \gamma_{(2)}) + (\alpha \circ \gamma) \hat{\mathbf{A}} \beta. \tag{2.36}$$

Proof. According to the definition of the shuffle product, $\gamma \circ (\alpha \succ A \prec \beta)$ consists of a sum of terms, two of which correspond to shuffling of γ into α, respectively β. In addition, we have all possible terms obtained by splitting γ into two ordered parts and shuffling the first into α and the second into β. The result is precisely our first formula. The second formula is obtained in the same way[16]. □

3. Realization by partial sum calculus

Let $\mathcal{N} := \{I \subset \mathbb{N} | I \neq \emptyset, |I| < \infty\}$. This is the set of non-empty finite subsets of the set of natural numbers. Let $\mathcal{A}$ be the freely generated linear space (over $\mathbb{K}$) with basis $\{e_I | I \in \mathcal{N}\}$. For $I, J \in \mathcal{N}$ we define the following associative products:

$$e_I \prec e_J := \begin{cases} e_{I \cup J} & \text{if} \quad \max(I) < \min(J) \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

$$e_I \bullet e_J := \begin{cases} e_{I \cup J} & \text{if} \quad \max(I) = \min(J) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

and thus

$$e_I \succ e_J = \begin{cases} e_{I \cup J} & \text{if} \quad \max(I) \leqslant \min(J) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

For example, $e_{\{2,4,5\}} = e_2 \prec e_4 \prec e_5 \in \mathcal{A}^3$, where we simply write e_n instead of $e_{\{n\}}$. Any element A of $\mathcal{A}^1$ can be written as $A = \sum_{n \geqslant 1} a_n e_n$ with $a_n \in \mathbb{K}$. The $\bullet$-product with another element $B = \sum_{n \geqslant 1} b_n e_n$ of $\mathcal{A}^1$ is then given by

$$A \bullet B = \sum_{n \geqslant 1} a_n b_n e_n. \tag{3.4}$$

[16] The formulae of this proposition do *not* hold with $\hat{\mathbf{A}}$ and $\check{\mathbf{A}}$ exchanged, if $\circ$ is not commutative. For example, $B \circ (\alpha \hat{\mathbf{A}} \beta) = (B \circ \alpha) \prec A \succ \beta + \alpha \prec B \circ (A \succ \beta) - \alpha \prec B \prec A \succ \beta$ where the last term corrects a double counting of the first two. By use of (2.12), we find $B \circ (\alpha \hat{\mathbf{A}} \beta) = (B \circ \alpha) \prec A \succ \beta + \alpha \prec A \succ (B \circ \beta) + \alpha \prec [B, A]_\circ \succ \beta$.

There is a formal[17] unit element, $E := \sum_{n\geqslant 1} e_n$. With $A_i = \sum_{n\geqslant 1} a_{i,n} e_n$, $i = 1, \ldots, r$, we obtain

$$A_1 \prec \cdots \prec A_r = \sum_{1\leqslant n_1<\cdots<n_r} a_{1,n_1} \cdots a_{r,n_r} e_{\{n_1,\ldots,n_r\}}. \tag{3.5}$$

For the main product, we find the simple formula

$$e_I \circ e_J = e_{I\cup J}. \tag{3.6}$$

The linear map $\Sigma_N : \mathcal{A} \to \mathbb{K}$ defined by

$$\Sigma_N(e_I) = \begin{cases} 1 & \text{if} \quad I \subset \{1, 2, \ldots, N\} \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

has the properties

$$\Sigma_N(A_1 \prec \cdots \prec A_r) = \sum_{1\leqslant n_1<\cdots<n_r\leqslant N} a_{1,n_1} \cdots a_{r,n_r} \tag{3.8}$$

$$\Sigma_N(A_1 \circ \cdots \circ A_r) = \left(\sum_{n_1=1}^{N} a_{1,n_1}\right) \cdots \left(\sum_{n_r=1}^{N} a_{1,n_r}\right). \tag{3.9}$$

By application of Σ_N to identities in (the partial sum realization of) the algebra $\mathcal{A}$, we obtain sum identities of the kind considered in the introduction, which hold for all N. But which identities in $\mathcal{A}$ correspond to the equations of the (x)ncKP hierarchy? The answer will be given in section 7.

Remark. The calculus of partial sums is known to carry the structure of a Rota–Baxter algebra [14, 15] (see also appendix A). We define a map R from $\mathcal{A}$ to a completion (as a projective limit) $\bar{\mathcal{A}}^1$ of $\mathcal{A}^1$ by

$$R(\alpha) = \sum_{N\geqslant 1} \Sigma_{N-1}(\alpha) e_N \qquad \forall \alpha \in \mathcal{A} \tag{3.10}$$

where $\Sigma_0(\alpha) := 0$. It satisfies

$$R(\alpha \prec A) = R(R(\alpha) \bullet A) \tag{3.11}$$

and therefore

$$R(A_1 \prec \cdots \prec A_r) = R(R(\ldots R(R(A_1) \bullet A_2) \bullet \ldots) \bullet A_r) \tag{3.12}$$

for $A_1, \ldots, A_r \in \mathcal{A}^1$. Another simple consequence of (3.11) is

$$R(\alpha \succ A) = R(R(\alpha) \bullet A + \alpha \bullet A). \tag{3.13}$$

Furthermore, for all $\alpha, \beta \in \mathcal{A}$ the following identity holds:

$$R(\alpha \circ \beta) = R(\alpha) \bullet R(\beta) \qquad \forall \alpha, \beta \in \mathcal{A}. \tag{3.14}$$

Applying R to $A \circ B = A \succ B + B \prec A$ thus leads to

$$R(A) \bullet R(B) = R(R(A) \bullet B + A \bullet R(B) + A \bullet B) \qquad \forall A, B \in \mathcal{A}^1. \tag{3.15}$$

With obvious extensions of $\bullet$ and R, $(\bar{\mathcal{A}}^1, \bullet, R|_{\bar{\mathcal{A}}^1})$ becomes a Rota–Baxter algebra of weight -1.

[17] Proper elements of $\mathcal{A}$ are *finite* sums.

4. The subalgebra of $\mathcal{A}$ generated by a single element P

Let $\mathcal{A}(P)$ be the subalgebra of $\mathcal{A}$ generated by an element $P \in \mathcal{A}^1$. More precisely, if $(\mathcal{A}(P), \bullet)$ has a unit element E, then $\mathcal{A}^1(P)$ is spanned by

$$P_n := P^{\bullet n} \qquad n = 0, 1, 2, \ldots \tag{4.1}$$

where $P_0 := E$. If $(\mathcal{A}(P), \bullet)$ is not unital, we have to disregard expressions containing P_0 in the following. Clearly, $(\mathcal{A}^1(P), \bullet)$ is commutative, and thus also $(\mathcal{A}(P), \circ)$ by theorem 2.1. According to section 2.3, P determines an associative product,

$$\alpha \hat{\times} \beta := -\alpha \hat{\mathbf{P}} \beta = -\alpha \prec P \succ \beta \qquad \forall \alpha, \beta \in \mathcal{A}(P) \tag{4.2}$$

which will play an important role in our subsequent considerations.

Proposition 4.1. *Via the main product, each $A \in \mathcal{A}^1(P)$ acts on a $\hat{\times}$-product according to the derivation rule*

$$A \circ (\alpha \hat{\times} \beta) = (A \circ \alpha) \hat{\times} \beta + \alpha \hat{\times} (A \circ \beta). \tag{4.3}$$

Proof. By use of (2.34), taking the commutativity of $(\mathcal{A}(P), \circ)$ into account. □

It is convenient to introduce the following objects which form a basis of $\mathcal{A}(P)$,

$$P_{m_1 \ldots m_k} := P_{m_1} \prec \cdots \prec P_{m_k}. \tag{4.4}$$

Theorem 4.1.

$$P_{m_1 \ldots m_k} \circ (\alpha \hat{\times} \beta) = \sum_{j=0}^{k} \left(P_{m_1 \ldots m_j} \circ \alpha\right) \hat{\times} \left(P_{m_{j+1} \ldots m_k} \circ \beta\right). \tag{4.5}$$

Proof. Since $\circ$ is commutative in the case under consideration, (2.36) implies

$$(A_1 \prec A_2 \prec \cdots \prec A_k) \circ (\alpha \hat{\times} \beta) = (A_1 \prec \cdots \prec A_k) \circ \alpha \hat{\times} \beta$$
$$+ \sum_{l=1}^{k-1} (A_1 \prec \cdots \prec A_l) \circ \alpha \hat{\times} (A_{l+1} \prec \cdots \prec A_k) \circ \beta + \alpha \hat{\times} (A_1 \prec \cdots \prec A_k) \circ \beta$$

for arbitrary $A_l \in \mathcal{A}^1$. Setting $A_l = P_{m_l}$ completes the proof. □

Remark. It looks natural to consider still another product: $\alpha \check{\times} \beta := \alpha \check{\mathbf{P}} \beta := \alpha \succ P \prec \beta$. Choosing the involution $^\psi$ in such a way that $P^\psi = P$, lemma (2.3) implies $(\alpha \check{\times} \beta)^\psi = -\alpha^\psi \hat{\times} \beta^\psi$. The product $\check{\times}$ is thus equivalent to the product $\hat{\times}$ and it is sufficient to deal with the latter, as long as we restrict our considerations to the algebra $\mathcal{A}(P)$.

4.1. Special relations in $\mathcal{A}(P)$ and reminiscences of (x)ncKP

The aim of this section is to derive algebraic identities in $\mathcal{A}(P)$ which mirror algebraic properties of the (x)ncKP hierarchy, as derived in [6]. The results will be important in later sections, where the relation between identities in $\mathcal{A}(P)$ and the ncKP hierarchy (and extensions) is put on firmer grounds.

Lemma 4.1.

$$P^{\circ n} = P \succ P^{\circ n-1} - \sum_{r=1}^{n-2} \binom{n-1}{r} P^{\circ n-r-1} \hat{\times} P^{\circ r} + P^{\circ n-1} \prec P \qquad n = 2, 3, \ldots \tag{4.6}$$

$$P^{\circ n-2} \circ (P \prec P) = P^{\circ n-1} \prec P - \sum_{r=1}^{n-2} \binom{n-2}{r} P^{\circ n-r-1} \,\hat{\times}\, P^{\circ r} \qquad n = 3, 4, \ldots. \tag{4.7}$$

Proof. For $n = 2$, the first relation obviously holds. Let us assume that the formula holds for some integer $n \geqslant 2$. Then

$$P^{\circ n+1} = P^{\circ n} \circ P = \left(P \succ P^{\circ n-1} - \sum_{r=1}^{n-2} \binom{n-1}{r} P^{\circ n-r-1} \,\hat{\times}\, P^{\circ r} + P^{\circ n-1} \prec P \right) \circ P.$$

Next we use (2.13), $P^{\circ 2} = P \succ P + P \prec P$, and (4.3) to obtain

$$\begin{aligned} P^{\circ n+1} = {} & P \succ P^{\circ n} + P^{\circ n} \prec P - P \,\hat{\times}\, P^{\circ n-1} - P^{\circ n-1} \,\hat{\times}\, P \\ & - \sum_{r=1}^{n-2} \binom{n-1}{r} (P^{\circ n-r-1} \,\hat{\times}\, P^{\circ r+1} + P^{\circ n-r} \,\hat{\times}\, P^{\circ r}). \end{aligned}$$

With the help of the combinatorial identity

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \tag{4.8}$$

and some simple manipulations, this becomes

$$P^{\circ n+1} = P \succ P^{\circ n} + P^{\circ n} \prec P - \sum_{r=1}^{n-1} \binom{n}{r} P^{\circ n-r} \,\hat{\times}\, P^{\circ r}$$

so that the first formula of the lemma also holds for $n + 1$. The proof of the second formula can be carried out in a very similar way. □

Let us introduce $U_2 := P$ and

$$U_n := (-1)^n P \prec P^{\circ n-2} \qquad n = 3, 4, \ldots \tag{4.9}$$

Proposition 4.2.

$$P \circ U_{n+1} = \frac{1}{2}(P_2 - P^{\circ 2}) \circ U_n - [U_2, U_n]_{\hat{\times}} + \sum_{r=1}^{n-2} \binom{n-2}{r} (-1)^r U_{n-r} \,\hat{\times}\, P^{\circ r} \circ U_2 \tag{4.10}$$

where $[\alpha, \beta]_{\hat{\times}} := \alpha \,\hat{\times}\, \beta - \beta \,\hat{\times}\, \alpha$.

Proof. First we note that, by use of (2.12), definition (4.9) implies $P \circ U_n = -U_{n+1} + P \succ U_n$ and, by multiple use of this equation,

$$P^{\circ 2} \circ U_n = P \circ (P \circ U_n) = -U_{n+2} - 2P \circ U_{n+1} - 2P \,\hat{\times}\, U_n + P_2 \succ U_n.$$

Furthermore, with the help of (2.12), $P_2 = P^{\circ 2} - 2P \prec P$, and (4.7), we obtain

$$\begin{aligned} P_2 \circ U_n &= (-1)^n P_2 \circ (P \prec P^{\circ n-2}) = P_2 \succ U_n + (-1)^n P \prec (P_2 \circ P^{\circ n-2}) \\ &= P_2 \succ U_n + U_{n+2} - 2(-1)^n P \prec ((P \prec P) \circ P^{\circ n-2}) \\ &= P_2 \succ U_n + U_{n+2} + 2U_{n+1} \prec P + 2(-1)^n P \prec \sum_{r=1}^{n-2} \binom{n-2}{r} P^{\circ n-r-1} \,\hat{\times}\, P^{\circ r} \\ &= P_2 \succ U_n + U_{n+2} + 2U_{n+1} \prec P - 2\sum_{r=1}^{n-2} \binom{n-2}{r} (-1)^r U_{n-r+1} \,\hat{\times}\, P^{\circ r}. \end{aligned}$$

Now we can eliminate the products $\prec$ and $\succ$ from this expression with the help of our first result and

$$U_{n+1} \prec P = -U_{n+2} + \sum_{r=1}^{n-1} \binom{n-1}{r} (-1)^r U_{n-r+1} \hat{\times} P^{\circ r}$$

which is obtained by applying $P \prec$ to (4.6). After simple manipulations and use of (4.8), this results in the desired formula. □

Next we introduce $H_1^{(m_1,\ldots,m_r)} := P_{m_r} \succ \cdots \succ P_{m_1}$ and

$$H_{n+1}^{(m_1,\ldots,m_r)} := H_n \succ P_{m_r} \succ \cdots \succ P_{m_1} \qquad n \in \mathbb{N} \tag{4.11}$$

where

$$H_n := H_n^{(1)} := P^{\succ n} \qquad n \in \mathbb{N}. \tag{4.12}$$

Proposition 4.3.

$$P_n \circ H_k^{(m)} - P_m \circ H_k^{(n)} - \sum_{j=1}^{k-1} \left[H_j^{(m)}, H_{k-j}^{(n)} \right]_{\hat{\times}} = 0. \tag{4.13}$$

Proof. Using (2.11) and (4.11), we obtain

$$\begin{aligned} H_{k-1} \circ P_m &= H_{k-1} \succ P_m + \sum_{j=1}^{k-2} H_j \succ P_m \prec H_{k-j-1} + P_m \prec H_{k-1} \\ &= H_k^{(m)} + \sum_{j=1}^{k-3} H_{j+1}^{(m)} \prec P \succ H_{k-j-2} + H_{k-1}^{(m)} \prec P + P_m \prec P \succ H_{k-2} \\ &= H_k^{(m)} - \sum_{j=1}^{k-2} H_j^{(m)} \hat{\times} H_{k-j-1} + H_{k-1}^{(m)} \prec P \end{aligned}$$

and thus

$$H_{k-1} \circ P_m \succ P_n = H_k^{(m)} \succ P_n - \sum_{j=1}^{k-1} H_j^{(m)} \hat{\times} H_{k-j}^{(n)}.$$

This is used to derive

$$\begin{aligned} P_m \circ H_k^{(n)} &= P_m \circ (P^{\succ k-1} \succ P_n) \\ &= (P_m \circ P^{\succ k-1}) \succ P_n + P^{\succ k-1} \succ (P_m \circ P_n) - H_k^{(m)} \succ P_n \\ &= H_{k-1} \succ (P_m \circ P_n) - \sum_{j=1}^{k-1} H_j^{(m)} \hat{\times} H_{k-j}^{(n)} \end{aligned}$$

from which (4.13) follows by anti-symmetrization with respect to m, n. □

Proposition 4.4.

$$H_n^{(m+1)} = -P_m \circ H_n^{(1)} + H_{n+1}^{(m)} + H_{m+n}^{(1)} - \sum_{r=1}^{n-1} H_{n-r}^{(m)} \hat{\times} H_r^{(1)} + \sum_{r=1}^{m-1} H_n^{(m-r)} \hat{\times} H_r^{(1)}. \tag{4.14}$$

Proof. First we obtain

$$P_m \succ P = H_{m+1} - \sum_{r=1}^{m-1} P_{m-r} \prec H_{r+1}$$

by induction on m. This shows that

$$P_m \succ H_n = H_{m+n} + \sum_{r=1}^{m-1} P_{m-r} \hat{\times} H_{n+r-1}$$

holds for $n = 1$, and the general formula is easily verified by induction on n. According to (2.10),

$$P_m \circ H_n = P_m \succ H_n + \sum_{r=1}^{n-1} H_r \prec P_m \succ H_{n-r} + H_n \prec P_m.$$

Using $H_n \prec P_m = H_{n+1}^{(m)} - H_n^{(m+1)}$, which is easily verified, this becomes

$$H_n^{(m+1)} - H_{n+1}^{(m)} + P_m \circ H_n = P_m \succ H_n + \sum_{r=1}^{n-1} H_r \prec P_m \succ H_{n-r}.$$

Now we eliminate all expressions $P_m \succ H_l$ by means of the corresponding formula above to get

$$\begin{aligned} H_n^{(m+1)} - H_{n+1}^{(m)} + P_m \circ H_n = H_{m+n} + \sum_{r=1}^{m-1} P_{m-r} \hat{\times} H_{n+r-1} \\ + \sum_{r=1}^{n-1} H_r \prec \left(H_{m+n-r} + \sum_{k=1}^{m-1} P_{m-k} \hat{\times} H_{n-r+k-1} \right). \end{aligned}$$

Next we use $H_r \prec P_{m-k} = H_{r+1}^{(m-k)} - H_r^{(m-k+1)}$. Some rearrangements then lead to (4.14). □

Proposition 4.5.

$$H_n^{(m_1,\ldots,m_{r+1})} = H_{n+m_{r+1}}^{(m_1,\ldots,m_r)} + \sum_{k=1}^{m_{r+1}-1} H_n^{(m_{r+1}-k)} \hat{\times} H_k^{(m_1,\ldots,m_r)} \qquad r = 1, 2, \ldots. \tag{4.15}$$

Proof. By induction one easily verifies that

$$P_n = H_n - \sum_{k=1}^{n-1} P_{n-k} \prec H_k.$$

Using this in definition (4.11), we find

$$\begin{aligned} H_n^{(m_1,\ldots,m_{r+1})} &= H_{n-1} \succ \left(H_{m_{r+1}} - \sum_{k=1}^{m_{r+1}-1} P_{m_{r+1}-k} \prec H_k \right) \succ P_{m_r} \succ \cdots \succ P_{m_1} \\ &= H_{n+m_{r+1}-1}^{(m_1,\ldots,m_r)} - \sum_{k=1}^{m_{r+1}-1} H_n^{(m_{r+1}-k)} \prec P \succ H_k^{(m_1,\ldots,m_r)} \end{aligned}$$

which is (4.15). □

Let $C_1^{(m_1,\ldots,m_r)} := (-1)^r P_{m_1\ldots m_r}$ and

$$C_{n+1}^{(m_1,\ldots,m_r)} := (-1)^{n+r} P_{m_1\ldots m_r} \prec P^{\prec n} \qquad n \in \mathbb{N}. \tag{4.16}$$

Proposition 4.6.

$$C_n^{(m+1)} = P_m \circ C_n^{(1)} + C_{n+1}^{(m)} + C_{m+n}^{(1)} + \sum_{r=1}^{n-1} C_r^{(1)} \,\hat{\times}\, C_{n-r}^{(m)} - \sum_{r=1}^{m-1} C_r^{(1)} \,\hat{\times}\, C_n^{(m-r)} \tag{4.17}$$

$$C_n^{(m_1,\ldots,m_{r+1})} = C_{n+m_{r+1}}^{(m_1,\ldots,m_r)} - \sum_{k=1}^{m_{r+1}-1} C_k^{(m_1,\ldots,m_r)} \,\hat{\times}\, C_n^{(m_{r+1}-k)}. \tag{4.18}$$

Proof. Choose the involution $^{\omega}$ such that $P^{\omega} = -P$. Then $P_r{}^{\omega} = -P_r$, $(\alpha \,\hat{\times}\, \beta)^{\omega} = -\beta^{\omega} \,\hat{\times}\, \alpha^{\omega}$, and $C_n^{(m_1,\ldots,m_r)} = \big(H_n^{(m_1,\ldots,m_r)}\big)^{\omega}$. Now our statements follow by application of $^{\omega}$ to (4.14) and (4.15). □

Let

$$A_{mn} := \tfrac{1}{2}(P_{mn} - P_{nm}) = \tfrac{1}{2}(P_m \prec P_n - P_n \prec P_m) = \tfrac{1}{2}(P_m \succ P_n - P_n \succ P_m). \tag{4.19}$$

Proposition 4.7.

$$A_{mn} \circ (\alpha \,\hat{\times}\, \beta) = A_{mn} \circ \alpha \,\hat{\times}\, \beta + \alpha \,\hat{\times}\, A_{mn} \circ \beta + \tfrac{1}{2}(P_m \circ \alpha \,\hat{\times}\, P_n \circ \beta - P_n \circ \alpha \,\hat{\times}\, P_m \circ \beta). \tag{4.20}$$

Proof. This follows directly from theorem 4.1. □

Proposition 4.8.

$$\begin{aligned} A_{mn} &= -\frac{1}{2}(P_{m+n} + P_m \circ P_n) + H_{m+1}^{(n)} + \sum_{r=1}^{m-1} P_r \,\hat{\times}\, H_{m-r}^{(n)} \\ &= -\frac{1}{2}(P_{m+n} - P_m \circ P_n) - C_{m+1}^{(n)} - \sum_{r=1}^{m-1} C_{m-r}^{(n)} \,\hat{\times}\, P_r. \end{aligned} \tag{4.21}$$

Proof. Using $P_m \circ P_n = P_m \succ P_n + P_n \prec P_m$ and $P_{m+n} = P_m \succ P_n - P_m \prec P_n$ we find

$$A_{mn} = P_m \succ P_n - \tfrac{1}{2}(P_m \circ P_n + P_{m+n}).$$

The first equality of the proposition now follows with the help of

$$P_m \succ P_n = H_{m+1}^{(n)} + \sum_{r=1}^{m-1} P_r \,\hat{\times}\, H_{m-r}^{(n)}$$

which is a special case of (4.15). The second equality is obtained by application of $^{\omega}$ to the first. □

Adding the two expressions for A_{mn} derived in the last proposition, leads to

$$A_{mn} = -\frac{1}{2}\left(P_{m+n} + C_{m+1}^{(n)} - H_{m+1}^{(n)} + \sum_{r=1}^{m-1} \big(C_{m-r}^{(n)} \,\hat{\times}\, P_r - P_r \,\hat{\times}\, H_{m-r}^{(n)}\big)\right) \tag{4.22}$$

and subtraction yields

$$P_m \circ P_n = C_{m+1}^{(n)} + H_{m+1}^{(n)} + \sum_{r=1}^{m-1} \big(C_{m-r}^{(n)} \,\hat{\times}\, P_r + P_r \,\hat{\times}\, H_{m-r}^{(n)}\big). \tag{4.23}$$

As a consequence of propositions 4.4–4.6 and some results of the following subsection (see (4.35) and (4.36)), the expressions $C_n^{(m_1,\ldots,m_r)}$ and $H_n^{(m_1,\ldots,m_r)}$ can be iteratively expressed completely in terms of only P_m, $m = 1, 2, \ldots$, the main product $\circ$ and the $\hat{\times}$-product. We will refer to this result in sections 7 and 8.

Equation (4.23) shows that the expressions constructed in this way are not all independent, but satisfy certain identities, and these actually correspond to ncKP equations. This correspondence will be firmly established in section 7. At this stage we already recognize it by comparing identities derived above with corresponding formulae in section 5 of [6], keeping the relations in the introduction and (3.9) in mind. In this way, the ncKP expression (5.31) in [6] for $\phi_{t_m t_n}$ finds its algebraic counterpart in (4.23), provided that $*$ corresponds to $\hat{\times}$. Such a (at this point still somewhat vague) correspondence is indeed observed between further (x)ncKP relations in [6] and algebraic identities in this section. The first non-trivial equation which arises from (4.23) is the one with $m = n = 2$ and yields

$$4P \circ P_3 - P^{\circ 4} - 3P_2 \circ P_2 = 6P \circ (P \hat{\times} P) - 6(P \hat{\times} P_2 - P_2 \hat{\times} P) \qquad (4.24)$$

which should be compared with (1.13) (see also the end of section 5.3).

Taking further algebraic objects built with $\prec$ into consideration, we obtain additional identities. With the choice $\{P_n, A_{mn}\}$ we have the identities (4.22) and a correspondence with xncKP equations is achieved (cf (5.30) in [6]). This will be made precise in section 7.2. Since the basis $\{P_{m_1\ldots m_k}\}$ of $\mathcal{A}(P)$ contains more objects, one should expect that an extension of the ncKP hierarchy exists which contains counterparts of all of them. This expectation will be confirmed in section 8.

Let us recall the underlying idea which might have got lost during the development of so much formalism. In the partial sum calculus realization, identities such as (4.23) become relations between sums where the summations run from 1 to some number $N \in \mathbb{N}$. The latter number is completely arbitrary, however. Hence we obtain families of sum identities if we let N run through the natural numbers. Mapping the original identities in $\mathcal{A}(P)$ properly to partial differential equations, as sketched in the introduction, the resulting differential equations will be solvable by the ansatz (1.7) and thus admit KP-like soliton solutions.

4.2. Symmetric functions

A simple calculation yields

$$\begin{aligned}
P_m \circ H_n &= P_m \succ H_n + \sum_{r=1}^{n-1} H_r \prec P_m \succ H_{n-r} + H_n \prec P_m \\
&= P_m \succ H_n + \sum_{r=2}^{n-1} H_{r-1} \succ (P \succ P_m - P_{m+1}) \succ H_{n-r} \\
&\quad + (P \succ P_m - P_{m+1}) \succ H_{n-1} + H_n \prec P_m \\
&= P_m \succ H_n + \sum_{r=1}^{n-1} H_r \succ P_m \succ H_{n-r} - \sum_{r=1}^{n-2} H_r \succ P_{m+1} \succ H_{n-r-1} \\
&\quad - P_{m+1} \succ H_{n-1} + H_n \succ P_m - H_{n-1} \succ P_{m+1}.
\end{aligned} \qquad (4.25)$$

Summing this relation properly, we obtain

$$nH_n = \sum_{r=1}^{n} P_r \circ H_{n-r} \qquad n \in \mathbb{N}. \qquad (4.26)$$

A similar calculation, or a simple application of the involution $^\psi$ to the last formula[18], leads to

$$nC_n = \sum_{r=1}^{n} (-1)^{r-1} C_{n-r} \circ P_r \qquad n \in \mathbb{N} \tag{4.27}$$

where

$$C_n := P^{\prec n} = (-1)^n C_n^{(1)} \qquad n \in \mathbb{N}. \tag{4.28}$$

Defining generating functions (with an indeterminate λ) by

$$H(\lambda) := \sum_{n\geqslant 0} H_n \lambda^n \qquad C(\lambda) := \sum_{n\geqslant 0} C_n \lambda^n \qquad P(\lambda) := \sum_{n\geqslant 1} P_n \lambda^{n-1} \tag{4.29}$$

where $H_0 = C_0 = I$ with a unit[19] I of the $\circ$-product, allows us to express (4.26) and (4.27) in the form

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} H(\lambda) = P(\lambda) \circ H(\lambda) \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} C(\lambda) = P(-\lambda) \circ C(\lambda). \tag{4.30}$$

Setting

$$\tilde{P}(\lambda) := \int P(\lambda)\, \mathrm{d}\lambda = \sum_{n\geqslant 1} \frac{P_n}{n} \lambda^n \tag{4.31}$$

we find

$$H(\lambda) = \mathrm{e}_\circ^{\tilde{P}(\lambda)} \qquad C(\lambda) = \mathrm{e}_\circ^{-\tilde{P}(-\lambda)} \tag{4.32}$$

where the exponentials are built with the $\circ$-product. This implies $C(-\lambda) \circ H(\lambda) = I$ and thus

$$\sum_{r=0}^{n} (-1)^r C_r \circ H_{n-r} = 0. \tag{4.33}$$

Moreover, recalling the definition

$$\mathrm{e}^{\sum_{n\geqslant 1} x_n \lambda^n} = \sum_{n\geqslant 0} \chi_n(x_1, x_2, x_3, \ldots) \lambda^n \tag{4.34}$$

(with commuting variables x_k, $k = 1, 2, \ldots$) of the *Schur polynomials*, we obtain

$$H_n = \chi_n(P, P_2/2, P_3/3, \ldots) = \sum_{|\mu|=n} z_\mu^{-1} P_1^{\circ m_1} \circ \cdots \circ P_n^{\circ m_n} \tag{4.35}$$

$$\begin{aligned} C_n &= (-1)^n \chi_n(-P, -P_2/2, -P_3/3, \ldots) \\ &= (-1)^n \sum_{|\mu|=n} z_\mu^{-1} (-1)^{m_1+\cdots+m_n} P_1^{\circ m_1} \circ \cdots \circ P_n^{\circ m_n} \end{aligned} \tag{4.36}$$

where the sum is over all partitions $\mu = (1^{m_1} 2^{m_2} \ldots n^{m_n})$ of n (so that $n = m_1 1 + m_2 2 + \cdots + m_n n$ with $m_r \in \mathbb{N} \cup \{0\}$), and

$$z_\mu := \prod_{r=1}^{n} r^{m_r} m_r!. \tag{4.37}$$

Writing $P = \sum_{k\geqslant 1} p_k e_k$ in the case of the partial sum calculus,

$$\Sigma_N(P_n) = \sum_{k=1}^{N} p_k^n \tag{4.38}$$

[18] If we choose the involution $^\psi$ such that $P^\psi = P$, then $P_n{}^\psi = (-1)^{n-1} P_n$ and $C_n{}^\psi = H_n$.

[19] Here the unit element I is only introduced temporarily in order to achieve compact expressions in terms of the exponential function.

is the nth *power sum*,

$$\Sigma_N(C_n) = \sum_{1\leqslant k_1<\cdots<k_n\leqslant N} p_{k_1}\cdots p_{k_n} \tag{4.39}$$

the nth *elementary symmetric polynomial*, and

$$\Sigma_N(H_n) = \sum_{1\leqslant k_1\leqslant\cdots\leqslant k_n\leqslant N} p_{k_1}\cdots p_{k_n} \tag{4.40}$$

the *complete symmetric polynomial* of degree n in N indeterminates $p_1, \ldots, p_N$ [35].

Remark. Applying the Rota–Baxter operator R defined in (3.10) to $C(\lambda)$, using (4.28), (4.29), (4.32) and (3.12), leads to

$$R\big(\mathrm{e}_\circ^{-\tilde{P}(-\lambda)}\big) = \sum_{n\geqslant 0} \lambda^n R(R(\cdots R(R(P)\bullet P)\bullet\cdots)\bullet P). \tag{4.41}$$

On the other hand, according to (3.14) we have

$$R\big(\mathrm{e}_\circ^{-\tilde{P}(-\lambda)}\big) = \mathrm{e}_\bullet^{-R(\tilde{P}(-\lambda))}. \tag{4.42}$$

With the help of $\ln(1+x) = -\sum_{n\geqslant 1}(-1)^n x^n/n$, we can write

$$\tilde{P}(-\lambda) = \sum_{n\geqslant 1}(-1)^n P^{\bullet n}\lambda^n/n = -\ln_\bullet(1+\lambda P). \tag{4.43}$$

Hence

$$\sum_{n\geqslant 0}\lambda^n R(R(\cdots R(R(P)\bullet P)\bullet\cdots)\bullet P) = \exp_\bullet(-R(\ln_\bullet(1+\lambda P))) \tag{4.44}$$

which is the famous *Spitzer's formula* [14, 16, 36–38].

5. Embedding of $\mathcal{A}(P)$ into an algebra generated by two elements

In the previous section, we suggested a correspondence between identities in $\mathcal{A}(P)$ and the ncKP hierarchy (and certain extensions). Writing $P = \sum_{n\geqslant 1} p_n e_n$ in the partial sum realization and taking a look at the algebraic identities presented in the introduction, one immediately concludes that a second element $Q = \sum_{n\geqslant 1} q_n e_n$ is required. But in this section we show that it is actually sufficient to restrict considerations to $\mathcal{A}(P)$. This covers an important aspect of our framework (see also the conclusions). The material of the present section is, however, not used in the following sections.

In the following, $(\mathcal{A}(P), \bullet)$ will *not* be regarded as unital, i.e., we exclude a possible unit element E. It is convenient (though not necessary) to augment the algebra $\mathcal{A}$ by a new element I. The necessary preparations are presented in the next two subsections. The third subsection presents the main result, namely the existence of an 'embedding' Ψ of $\mathcal{A}(P)$ into an algebra generated by two elements P, Q such that certain homomorphism properties hold. The last subsection contains supplementary material (a generalization of symmetric functions).

5.1. The augmented algebra $\tilde{\mathcal{A}}$

The new element I will be required to satisfy

$$I \prec \alpha = \alpha = \alpha \prec I \qquad I \succ \alpha = \alpha = \alpha \succ I \qquad I \circ \alpha = \alpha = \alpha \circ I \tag{5.1}$$

which implies

$$\alpha \bullet I = I \bullet \alpha = 0. \tag{5.2}$$

A further consequence is

$$(\alpha \succ I) \prec \beta = \alpha \prec \beta \qquad \alpha \succ (I \prec \beta) = \alpha \succ \beta \tag{5.3}$$

which shows that we are forced to give up associativity in these particular combinations.

The *augmented algebra* $\tilde{\mathcal{A}}$ is again a graded algebra, with $\tilde{\mathcal{A}}^0 = \tilde{\mathcal{A}}_0 = \mathbb{K} I$ and $\tilde{\mathcal{A}} = \bigoplus_{r\geqslant 0} \tilde{\mathcal{A}}^r = \bigoplus_{r\geqslant 0} \tilde{\mathcal{A}}_r$ where $\tilde{\mathcal{A}}^r \simeq \mathcal{A}^r$, $\tilde{\mathcal{A}}_r \simeq \mathcal{A}_r$ for $r \geqslant 1$.

With each $A \in \tilde{\mathcal{A}}^1$ we associate products via (2.30) and (2.31) which are essentially[20] combined associative with all other products defined so far, with the exception of the main product, and thus also among themselves. In particular, we have

$$(\alpha \check{\mathbf{A}} \beta)\hat{\mathbf{B}}\gamma = \alpha \check{\mathbf{A}}(\beta \hat{\mathbf{B}}\gamma) \qquad (\alpha \hat{\mathbf{A}} \beta)\check{\mathbf{B}}\gamma = \alpha \hat{\mathbf{A}}(\beta \check{\mathbf{B}}\gamma) \tag{5.4}$$

for all $A, B \in \tilde{\mathcal{A}}^1$, $\alpha, \beta, \gamma \in \tilde{\mathcal{A}}$, and

$$\left.\begin{aligned} (\alpha \hat{\mathbf{A}} \beta)\hat{\mathbf{B}}\gamma &= \alpha \hat{\mathbf{A}}(\beta \hat{\mathbf{B}}\gamma) \\ (\alpha \check{\mathbf{A}} \beta)\check{\mathbf{B}}\gamma &= \alpha \check{\mathbf{A}}(\beta \check{\mathbf{B}}\gamma) \end{aligned}\right. \qquad \text{if} \quad \beta \neq I \tag{5.5}$$

so that we are allowed to drop the brackets and simply write, e.g., $\alpha \hat{\mathbf{A}} \beta \hat{\mathbf{B}} \gamma$ if $\beta \neq I$. Since

$$I\hat{\mathbf{A}}I = A \qquad I\hat{\mathbf{A}}\alpha = A \succ \alpha \qquad \alpha\hat{\mathbf{A}}I = \alpha \prec A \tag{5.6}$$

$$I\check{\mathbf{A}}I = A \qquad I\check{\mathbf{A}}\alpha = A \prec \alpha \qquad \alpha\check{\mathbf{A}}I = \alpha \succ A \tag{5.7}$$

we can express any element of $\tilde{\mathcal{A}}$ in terms of these operators. For example,

$$\begin{aligned} A_1 \succ A_2 \prec A_3 \prec A_4 \succ A_5 &= (A_1 \succ A_2 \prec A_3)\hat{\mathbf{A}}_4 A_5 \\ &= ((A_1 \succ A_2)\hat{\mathbf{A}}_3 I)\hat{\mathbf{A}}_4(I\hat{\mathbf{A}}_5 I) = ((I\hat{\mathbf{A}}_1(I\hat{\mathbf{A}}_2 I))\hat{\mathbf{A}}_3 I)\hat{\mathbf{A}}_4(I\hat{\mathbf{A}}_5 I) \\ &= (I\hat{\mathbf{A}}_1(I\hat{\mathbf{A}}_2 I)\hat{\mathbf{A}}_3 I)\hat{\mathbf{A}}_4(I\hat{\mathbf{A}}_5 I) =: I\hat{\mathbf{A}}_1(I\hat{\mathbf{A}}_2 I)(\hat{\mathbf{A}}_3 I)\hat{\mathbf{A}}_4(I\hat{\mathbf{A}}_5 I) \end{aligned}$$

where we introduced a simplified notation in the last step. The remaining brackets take care of the non-associativity of certain products with I. In the same way we get

$$A_1 \succ A_2 \prec A_3 \prec A_4 \succ A_5 = (I\check{\mathbf{A}}_1 I)\check{\mathbf{A}}_2(I\check{\mathbf{A}}_3)(I\check{\mathbf{A}}_4 I)\check{\mathbf{A}}_5 I.$$

Eliminating the I at both ends, we obtain two linear maps, $\alpha \mapsto \hat{\alpha}$, respectively $\alpha \mapsto \check{\alpha}$. In particular,

$$\begin{aligned} A_1 \succ A_2 \prec A_3 \prec A_4 \succ A_5 &\overset{\wedge}{\mapsto} \hat{\mathbf{A}}_1(I\hat{\mathbf{A}}_2 I)(\hat{\mathbf{A}}_3 I)\hat{\mathbf{A}}_4(I\hat{\mathbf{A}}_5) \\ A_1 \succ A_2 \prec A_3 \prec A_4 \succ A_5 &\overset{\vee}{\mapsto} (\check{\mathbf{A}}_1 I)\check{\mathbf{A}}_2(I\check{\mathbf{A}}_3)(I\check{\mathbf{A}}_4 I)\check{\mathbf{A}}_5. \end{aligned}$$

The following properties are quite evident:

$$\begin{aligned} \alpha \prec \beta &\overset{\wedge}{\mapsto} (\hat{\alpha} I)\hat{\beta} \qquad & \alpha \succ \beta &\overset{\wedge}{\mapsto} \hat{\alpha}(I\hat{\beta}) \\ \alpha \prec \beta &\overset{\vee}{\mapsto} \check{\alpha}(I\check{\beta}) \qquad & \alpha \succ \beta &\overset{\vee}{\mapsto} (\check{\alpha} I)\check{\beta}. \end{aligned}$$

The identities (2.10) and (2.11) can be written as

$$A \circ \alpha = I\hat{\mathbf{A}}\alpha + \sum \alpha_{(1)}\hat{\mathbf{A}}\alpha_{(2)} + \alpha\hat{\mathbf{A}}I =: \sum{}' \alpha_{(1)}\hat{\mathbf{A}}\alpha_{(2)} \tag{5.8}$$

$$\alpha \circ A = I\check{\mathbf{A}}\alpha + \sum \alpha_{(1)}\check{\mathbf{A}}\alpha_{(2)} + \alpha\check{\mathbf{A}}I =: \sum{}' \alpha_{(1)}\check{\mathbf{A}}\alpha_{(2)}. \tag{5.9}$$

[20] Non-associativity only appears in special expressions involving I.

5.2. The augmented subalgebra $\tilde{\mathcal{A}}(P)$

Let $\tilde{\mathcal{A}}(P)$ be the subalgebra of $\tilde{\mathcal{A}}$ obtained from the algebra $\mathcal{A}(P)$, which is generated by a single element $P \in \mathcal{A}^1$, by augmenting it with I. Then

$$P_n = I\hat{\mathbf{P}}_n I \qquad n \in \mathbb{N}. \tag{5.10}$$

Clearly, $(\tilde{\mathcal{A}}^1(P), \bullet)$ is commutative, and thus also $(\tilde{\mathcal{A}}(P), \circ)$ according to theorem 2.1.

Lemma 5.1. *The following identities hold for all $n \in \mathbb{N}$,*

$$\hat{\mathbf{P}}_{n+1} = \hat{\mathbf{P}}(I\hat{\mathbf{P}}_n) - (\hat{\mathbf{P}}I)\hat{\mathbf{P}}_n \tag{5.11}$$

$$\check{\mathbf{P}}_{n+1} = (\check{\mathbf{P}}I)\check{\mathbf{P}}_n - \check{\mathbf{P}}(I\check{\mathbf{P}}_n). \tag{5.12}$$

Proof.

$$\begin{aligned}\alpha\hat{\mathbf{P}}_{n+1}\beta &= \alpha \prec (P \succ P_n - P \prec P_n) \succ \beta = \alpha\hat{\mathbf{P}}(P_n \succ \beta) - \alpha \prec (P\hat{\mathbf{P}}_n\beta)\\ &= \alpha\hat{\mathbf{P}}(I\hat{\mathbf{P}}_n)\beta - (\alpha\hat{\mathbf{P}}I)\hat{\mathbf{P}}_n\beta.\end{aligned}$$

The second identity is verified in the same way. □

The product $\hat{\times}$ introduced in section 4, extended to $\tilde{\mathcal{A}}$, is essentially associative:

$$(\alpha \mathbin{\hat{\times}} \beta) \mathbin{\hat{\times}} \gamma = \alpha \mathbin{\hat{\times}} (\beta \mathbin{\hat{\times}} \gamma) \qquad \forall \alpha, \beta, \gamma \in \tilde{\mathcal{A}}(P), \quad \beta \neq I. \tag{5.13}$$

Note that

$$I \mathbin{\hat{\times}} I = -P \qquad I \mathbin{\hat{\times}} \alpha = -P \succ \alpha \qquad \alpha \mathbin{\hat{\times}} I = -\alpha \prec P. \tag{5.14}$$

By iterative use of (5.11), we can express $\hat{\mathbf{P}}_n$ in terms of only I and the product $\hat{\times}$. For example,

$$\begin{aligned}\alpha\hat{\mathbf{P}}_2\beta &= \alpha\hat{\mathbf{P}}(I\hat{\mathbf{P}})\beta - \alpha(\hat{\mathbf{P}}I)\hat{\mathbf{P}}\beta = -\alpha \mathbin{\hat{\times}} (I\hat{\mathbf{P}}\beta) + (\alpha\hat{\mathbf{P}}I) \mathbin{\hat{\times}} \beta\\ &= \alpha \mathbin{\hat{\times}} (I \mathbin{\hat{\times}} \beta) - (\alpha \mathbin{\hat{\times}} I) \mathbin{\hat{\times}} \beta.\end{aligned} \tag{5.15}$$

Since we have

$$P_m \prec P_n = P_m\hat{\mathbf{P}}_n I \qquad P_m \succ P_n = I\hat{\mathbf{P}}_m P_n \tag{5.16}$$

and similar formulae for expressions of higher grade, it follows that the algebraic structure of $\tilde{\mathcal{A}}(P)$ can be expressed completely in terms of the element I and the product $\hat{\times}$. Further examples of expressions in terms of I are $C_n = I(\hat{\mathbf{P}}I)^n$, $H_n = (I\hat{\mathbf{P}})^n I$,

$$P_{m_1 \ldots m_k} = \big(I\hat{\mathbf{P}}_{m_1}I\big)\big(\hat{\mathbf{P}}_{m_2}I\big)\cdots\big(\hat{\mathbf{P}}_{m_k}I\big) \tag{5.17}$$

and (cf (5.8))

$$P_n \circ P_{m_1 \ldots r_k} = \sum_{l=0}^{k} \big(\big(I\hat{\mathbf{P}}_{m_1}I\big)\cdots\big(\hat{\mathbf{P}}_{m_l}I\big)\big)\hat{\mathbf{P}}_n\big(\big(I\hat{\mathbf{P}}_{m_{l+1}}I\big)\cdots\big(\hat{\mathbf{P}}_{m_k}I\big)\big). \tag{5.18}$$

5.3. The embedding

Let $\tilde{\mathcal{A}}(P, Q)$ denote the subalgebra of $\tilde{\mathcal{A}}$ generated by two fixed elements $P, Q \in \mathcal{A}^1$ with the property $P \bullet Q = Q \bullet P$, so that $(\tilde{\mathcal{A}}^1, \bullet)$ is commutative and then, by theorem 2.1, also $(\tilde{\mathcal{A}}(P, Q), \circ)$. Let us introduce the product

$$\alpha \times \beta := -\alpha \mathbf{T} \beta = \alpha \succ Q \prec \beta - \alpha \prec P \succ \beta \tag{5.19}$$

where

$$\mathbf{T} := \hat{\mathbf{P}} - \check{\mathbf{Q}}. \tag{5.20}$$

The product $\times$ is essentially associative, i.e.,

$$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \tag{5.21}$$

for all $\alpha, \beta, \gamma \in \tilde{\mathcal{A}}(P, Q)$ with $\beta \neq I$.

Next we define a linear map $\Psi : \tilde{\mathcal{A}}(P) \to \tilde{\mathcal{A}}(P, Q)$ by $\Psi(I) = I$ and the homomorphism property

$$\Psi(\alpha \,\hat{\times}\, \beta) = \Psi(\alpha) \times \Psi(\beta) \qquad \forall \alpha, \beta \in \tilde{\mathcal{A}}(P). \tag{5.22}$$

Since I generates $\tilde{\mathcal{A}}(P)$ using the product $\hat{\times}$, this defines Ψ on $\tilde{\mathcal{A}}(P)$. In particular, it leads to

$$\Psi(P) = -\Psi(I \,\hat{\times}\, I) = -\Psi(I) \times \Psi(I) = -I \times I = P - Q = I \mathbf{T} I \tag{5.23}$$

and

$$\Psi(P \succ \alpha) = I \mathbf{T} \Psi(\alpha) \qquad \Psi(\alpha \prec P) = \Psi(\alpha) \mathbf{T} I. \tag{5.24}$$

Resolving the definitions of the two products in (5.22), the homomorphism property of Ψ reads $\Psi(\alpha \hat{\mathbf{P}} \beta) = \Psi(\alpha) \mathbf{T} \Psi(\beta)$, which can be expressed in the short form $\Psi(\hat{\mathbf{P}}) = \mathbf{T}$.

Proposition 5.1.

$$\Psi(\hat{\mathbf{P}}_n) = \hat{\mathbf{P}}_n - \check{\mathbf{Q}}_n \qquad n = 1, 2, \ldots \tag{5.25}$$

where $\hat{\mathbf{P}}_n$ *and* $\check{\mathbf{Q}}_n$ *are determined by* $P_n = P^{\bullet n}$ *and* $Q_n = Q^{\bullet n}$*, respectively.*

Proof. By construction of Ψ, (5.25) holds for $n = 1$. Let us now assume that $\Psi(\hat{\mathbf{P}}_n) = \hat{\mathbf{P}}_n - \check{\mathbf{Q}}_n =: \mathbf{T}_n$ holds for fixed $n \in \mathbb{N}$. Then Ψ applied to (5.11) yields

$$\begin{aligned}
\Psi(\alpha \hat{\mathbf{P}}_{n+1} \beta) &= \Psi(\alpha \hat{\mathbf{P}}(I \hat{\mathbf{P}}_n) \beta) - \Psi(\alpha (\hat{\mathbf{P}} I) \hat{\mathbf{P}}_n \beta) \\
&= -\Psi(\alpha) \times \Psi(I \hat{\mathbf{P}}_n \beta) - \Psi(\alpha \hat{\mathbf{P}} I) \mathbf{T}_n \Psi(\beta) \\
&= \Psi(\alpha) \mathbf{T} (\Psi(I) \mathbf{T}_n \Psi(\beta)) - (\Psi(\alpha) \mathbf{T} I) \mathbf{T}_n \Psi(\beta) \\
&= \Psi(\alpha) (\mathbf{T}(I \mathbf{T}_n) - (\mathbf{T} I) \mathbf{T}_n) \Psi(\beta).
\end{aligned}$$

Making use of (5.4), (5.11) and (5.12), we find

$$\mathbf{T}(I \mathbf{T}_n) - (\mathbf{T} I) \mathbf{T}_n = \hat{\mathbf{P}}(I \hat{\mathbf{P}}_n) - (\hat{\mathbf{P}} I) \hat{\mathbf{P}}_n + \check{\mathbf{Q}}(I \check{\mathbf{Q}}_n) - (\check{\mathbf{Q}} I) \check{\mathbf{Q}}_n = \hat{\mathbf{P}}_{n+1} - \check{\mathbf{Q}}_{n+1}.$$

Hence $\Psi(\hat{\mathbf{P}}_{n+1}) = \hat{\mathbf{P}}_{n+1} - \check{\mathbf{Q}}_{n+1}$ which completes the induction. □

Resolving the definitions involved, (5.25) reads

$$\Psi(\alpha \prec P_n \succ \beta) = \Psi(\alpha) \prec P_n \succ \Psi(\beta) - \Psi(\alpha) \succ Q_n \prec \Psi(\beta). \tag{5.26}$$

In particular, we obtain

$$\begin{aligned}
\Psi(\alpha \prec P_n) &= \Psi(\alpha) \prec P_n - \Psi(\alpha) \succ Q_n \\
\Psi(P_n \succ \beta) &= P_n \succ \Psi(\beta) - Q_n \prec \Psi(\beta).
\end{aligned} \tag{5.27}$$

Theorem 5.1. *The map Ψ defined above is a main product homomorphism, i.e.,*

$$\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta) \qquad \forall \alpha, \beta \in \tilde{\mathcal{A}}(P). \tag{5.28}$$

Proof. First we prove this property for $\alpha, \beta \in \tilde{\mathcal{A}}^1(P)$. It is sufficient to consider

$$\begin{aligned}\Psi(P_r \circ P_s) &= \Psi(P_r \succ P_s + P_s \prec P_r) = \Psi(I\hat{\mathbf{P}}_r P_s + P_s \hat{\mathbf{P}}_r I) = I\mathbf{T}_r T_s + T_s \mathbf{T}_r I \\ &= P_r \circ (P_s - Q_s) - (P_s - Q_s) \circ Q_r = (P_r - Q_r) \circ (P_s - Q_s)\end{aligned}$$

where $\mathbf{T}_r := \hat{\mathbf{P}}_r - \check{\mathbf{Q}}_r, T_r := P_r - Q_r$ (in deviation from our previous notation), and we used the commutativity of the main product in the last step. Hence $\Psi(P_r \circ P_s) = T_r \circ T_s = \Psi(P_r) \circ \Psi(P_s)$ and thus $\Psi(A \circ B) = \Psi(A) \circ \Psi(B)$ for all $A, B \in \tilde{\mathcal{A}}^1(P)$.

Next, we show that $\Psi(A \circ \beta) = \Psi(A) \circ \Psi(\beta), \forall \beta \in \tilde{\mathcal{A}}(P)$. It suffices to consider $A = P_n$ and

$$\beta = P_{m_1} \prec \cdots \prec P_{m_k} = \big(I\hat{\mathbf{P}}_{m_1} I\big)\big(\hat{\mathbf{P}}_{m_2} I\big) \cdots \big(\hat{\mathbf{P}}_{m_k} I\big).$$

By use of (5.18) and proposition 5.1, we find

$$\Psi(P_n \circ \beta) = \sum{}' \Psi(\beta_{(1)} \hat{\mathbf{P}}_n \beta_{(2)}) = \sum{}' \Psi(\beta_{(1)}) \mathbf{T}_n \Psi(\beta_{(2)}).$$

Iterated application of proposition 5.1 leads to

$$\begin{aligned}\Psi\big(P_{m_1} \prec \cdots \prec P_{m_j}\big) &= \Psi\big(\big(I\hat{\mathbf{P}}_{m_1} I\big) \cdots \big(\hat{\mathbf{P}}_{m_j} I\big)\big) = \big(I\mathbf{T}_{m_1} I\big) \cdots \big(\mathbf{T}_{m_j} I\big) \\ \Psi\big(P_{m_{j+1}} \prec \cdots \prec P_{m_k}\big) &= \Psi\big(\big(I\hat{\mathbf{P}}_{m_{j+1}} I\big) \cdots \big(\hat{\mathbf{P}}_{m_k} I\big)\big) = \big(I\mathbf{T}_{m_{j+1}} I\big) \cdots \big(\mathbf{T}_{m_k} I\big)\end{aligned}$$

and

$$\Psi(\beta) = \big(I\mathbf{T}_{m_1} I\big) \cdots \big(\mathbf{T}_{m_j} I\big)\big(\mathbf{T}_{m_{j+1}} I\big) \cdots \big(\mathbf{T}_{m_k} I\big)$$

so that

$$\Psi(\beta_{(1)}) = \Psi(\beta)_{(1)} \qquad \Psi(\beta_{(2)}) = \Psi(\beta)_{(2)}.$$

It follows that

$$\begin{aligned}\Psi(P_n \circ \beta) &= \sum{}' \Psi(\beta)_{(1)} \mathbf{T}_n \Psi(\beta)_{(2)} \\ &= \sum{}' \Psi(\beta)_{(1)} \hat{\mathbf{P}}_n \Psi(\beta)_{(2)} - \sum{}' \Psi(\beta)_{(1)} \check{\mathbf{Q}}_n \Psi(\beta)_{(2)} \\ &= P_n \circ \Psi(\beta) - \Psi(\beta) \circ Q_n = T_n \circ \Psi(\beta)\end{aligned}$$

where we used (2.35), (2.36), and again the commutativity of the main product in the last two steps. This implies $\Psi(A \circ \beta) = \Psi(A) \circ \Psi(\beta)$.

Finally, we prove our assertion in the general case by induction. We assume that it holds for all $\alpha \in \tilde{\mathcal{A}}^m(P)$ where $1 \leqslant m \leqslant n$ for fixed n, and all $\beta \in \tilde{\mathcal{A}}(P)$. The induction step is then carried out with the help of (2.17), i.e.,

$$(P_r \succ \alpha) \circ \beta = I\hat{\mathbf{P}}_r(\alpha \circ \beta) + \sum \beta_{(1)} \hat{\mathbf{P}}_r(\alpha \circ \beta_{(2)}) + \beta \hat{\mathbf{P}}_r \alpha = \sum{}' \beta_{(1)} \hat{\mathbf{P}}_r(\alpha \circ \beta_{(2)}).$$

Applying Ψ and using proposition 5.1, leads to

$$\begin{aligned}\Psi((P_r \succ \alpha) \circ \beta) &= \sum{}' \Psi(\beta_{(1)}) \mathbf{T}_r \Psi(\alpha \circ \beta_{(2)}) = \sum{}' \Psi(\beta)_{(1)} \mathbf{T}_r \Psi(\alpha) \circ \Psi(\beta)_{(2)} \\ &= \sum{}' \Psi(\beta)_{(1)} \hat{\mathbf{P}}_r \Psi(\alpha) \circ \Psi(\beta)_{(2)} - \sum{}' \Psi(\beta)_{(1)} \check{\mathbf{Q}}_r \Psi(\alpha) \circ \Psi(\beta)_{(2)} \\ &= (P_r \succ \Psi(\alpha)) \circ \Psi(\beta) - \Psi(\beta) \circ (Q_r \prec \Psi(\alpha)) \\ &= \Psi(P_r \succ \alpha) \circ \Psi(\beta)\end{aligned}$$

where we made use of (2.17), (2.18), and the commutativity of the $\circ$ product. This implies that our assertion also holds for $\alpha \in \tilde{\mathcal{A}}^{n+1}(P)$. $\square$

For generic Q, the map $\Psi : \tilde{\mathcal{A}}(P) \to \tilde{\mathcal{A}}(P, Q)$ is injective. It follows that Ψ is an isomorphism of the (double) algebras $(\mathcal{A}(P), \hat{\times}, \circ)$ and $(\mathcal{A}(P/Q), \times, \circ)$ where $\mathcal{A}(P/Q) := \Psi(\mathcal{A}(P))$.[21] Applying Ψ and afterwards Σ_N to the identity (4.24), for example, we recover the algebraic sum identity (1.13).

5.4. 'Supersymmetric' functions

Let us introduce

$$\tilde{T}(\lambda) := \sum_{n\geqslant 1} \frac{T_n}{n} \lambda^n := \tilde{P}(\lambda) - \tilde{Q}(\lambda) \tag{5.29}$$

where $\tilde{P}(\lambda)$ is given by (4.31) and $\tilde{Q}(\lambda)$ is defined in the same way (with P replaced by Q). Using the commutativity of $\circ$, we obtain

$$H^{P/Q}(\lambda) := \sum_{n\geqslant 0} H_n^{P/Q} \lambda^n := \mathrm{e}_\circ^{\tilde{T}(\lambda)} = \mathrm{e}_\circ^{\tilde{P}(\lambda)} \circ \mathrm{e}_\circ^{-\tilde{Q}(\lambda)} = H^P(\lambda) \circ C^Q(-\lambda) \tag{5.30}$$

where $H^P(\lambda)$ is given by the first of relations (4.32), and $C^Q(\lambda)$ by the second with P replaced by Q. Hence

$$H_n^{P/Q} = \sum_{r=0}^{n} (-1)^{n-r} H_r^P \circ C_{n-r}^Q. \tag{5.31}$$

In the same way, we obtain

$$C^{P/Q}(\lambda) := \sum_{n\geqslant 0} C_n^{P/Q} \lambda^n := \mathrm{e}_\circ^{-\tilde{T}(-\lambda)} \qquad C_n^{P/Q} = (-1)^n H_n^{Q/P}. \tag{5.32}$$

As a consequence of theorem 5.1,

$$H_n^{P/Q} = \Psi\big(H_n^P\big) \qquad C_n^{P/Q} = \Psi\big(C_n^P\big). \tag{5.33}$$

Using $P = \sum_{n\geqslant 1} p_n e_n$ and $Q = \sum_{n\geqslant 1} q_n e_n$ in the partial sum calculus, we obtain

$$\Sigma_N(T_r) = \sum_{k=1}^{N} \big(p_k^r - q_k^r\big). \tag{5.34}$$

A function $f(p_1, \ldots, p_N, q_1, \ldots, q_N)$ is called *doubly symmetric* if it is invariant under permutations of $p_1, \ldots, p_N$, as well as permutations of $q_1, \ldots, q_N$.[22] A doubly symmetric function is called *supersymmetric* if the substitution $q_1 = p_1$ results in a function which is independent of p_1 [40].[23] Together with 1, the sums (5.34) actually generate the algebra of supersymmetric polynomials of $N + N$ indeterminates [40]. Then $\Sigma_N\big(C_n^{P/Q}\big)$ and $\Sigma_N\big(H_n^{P/Q}\big)$ are the *elementary*, respectively the *complete supersymmetric polynomials* (see [42]).

6. From $\mathcal{A}(P)$ to the algebra of ΨDOs

In the following, $\mathcal{R}$ denotes the $\mathbb{K}$-algebra of formal pseudo-differential operators generated by a generic[24] L of the form (1.2) with the product $*$ and the projection $(\,)_{\geqslant 0}$. For $X, Y \in \mathcal{R}$,

$$\begin{aligned} X \vartriangle Y &:= X_{\geqslant 0} * Y_{\geqslant 0} - X_{<0} * Y_{<0} \\ &= X_{\geqslant 0} * Y - X * Y_{<0} = X * Y_{\geqslant 0} - X_{<0} * Y \end{aligned} \tag{6.1}$$

[21] From the construction of Ψ it is evident that the elements of $\mathcal{A}(P/Q)$ are invariant under simultaneous translations $P \mapsto P + A$, $Q \mapsto Q + A$ with any $A \in \tilde{\mathcal{A}}^1$ such that $A \bullet P = P \bullet A = 0$ and $A \bullet Q = Q \bullet A = 0$.
[22] Generalizations are sometimes called 'multi-symmetric functions', see [39] and the references cited therein.
[23] Such functions have been called *bisymmetric* in [41].
[24] In the sense that no non-trivial identities should hold in $\mathcal{R}$.

defines an associative product[25] $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$. As an immediate consequence of its definition, $\vartriangle$ has the property

$$\mathrm{res}(X \vartriangle Y) = 0 \qquad \forall X, Y \in \mathcal{R}. \tag{6.2}$$

In this section we consider $\mathcal{A}(P)$ *not* as unital, i.e., we exclude a possible unit element from the algebra. A corresponding extension is certainly possible, but not needed for our purposes. Let $\ell, r : \mathcal{A}(P) \to \mathcal{R}$ be the two linear maps defined iteratively by $\ell(P) = r(P) = L$ and

$$\ell(\alpha \prec P) = -\ell(\alpha)_{<0} * L \qquad \ell(\alpha \succ P) = \ell(\alpha)_{\geqslant 0} * L \tag{6.3}$$

$$r(P \prec \alpha) = -L * r(\alpha)_{\geqslant 0} \qquad r(P \succ \alpha) = L * r(\alpha)_{<0}. \tag{6.4}$$

The pseudo-differential operators defined by

$$L^{m_1,\ldots,m_k} := \ell\big(P_{m_1\ldots m_k}\big) \tag{6.5}$$

will be important in the following. In particular, they are used to define operators $\delta_{m_1\ldots m_k}$ in $\mathcal{R}$ iteratively by

$$\delta_{m_1\ldots m_k} L = -\big[L_{<0}^{m_1,\ldots,m_k}, L\big] + \sum_{j=1}^{k-1} \big(\delta_{m_1\ldots m_j} L\big) * L_{<0}^{m_{j+1},\ldots,m_k} \tag{6.6}$$

$$\delta_{m_1\ldots m_k}(X_{\geqslant 0}) = \big(\delta_{m_1\ldots m_k} X\big)_{\geqslant 0} \tag{6.7}$$

and the generalized derivation rule

$$\delta_{m_1\ldots m_k}(X * Y) = \sum_{j=0}^{k} \big(\delta_{m_1\ldots m_j} X\big) * \big(\delta_{m_{j+1}\ldots m_k} Y\big) \tag{6.8}$$

where $\delta_{m_1\ldots m_j} = \mathrm{id}$ if $j = 0$ and $\delta_{m_{j+1}\ldots m_k} = \mathrm{id}$ if $j = k$. We already met the simplest members δ_n of this family in the introduction, for which the last formula reduces to the ordinary derivation rule. After some preparations in the first two subsections, the third demonstrates that the $\delta_{m_1\ldots m_k}$ commute with each other. In the last subsection we explore properties of the linear map $\Phi : \mathcal{A}(P) \to \mathcal{R}_0$ defined by

$$\Phi(\alpha) := \mathrm{res}(\ell(\alpha)) \qquad \forall \alpha \in \mathcal{A}(P). \tag{6.9}$$

This map will play a crucial role in the following sections. The reader may jump from here directly to section 7 and skip the more technical subsections on first reading.

6.1. Properties of the maps ℓ and r

Lemma 6.1.

$$\ell(P_n) = L^n = r(P_n) \qquad n = 1, 2, \ldots. \tag{6.10}$$

Proof. Using the definition of ℓ, we find

$$\ell(P_{n+1}) = \ell(P_n \bullet P) = \ell(P_n \succ P) - \ell(P_n \prec P) = \ell(P_n)_{\geqslant 0} * L + \ell(P_n)_{<0} * L = \ell(P_n) * L.$$

Now the statement for ℓ follows by induction. The corresponding statement for the map r is obtained in the same way. □

[25] This product already appeared in [24]. It is an example of an associative product defined more generally in the framework of Rota–Baxter algebras, see appendix A. Indeed, $R(X) := X_{\geqslant 0}$ defines a Rota–Baxter operator on the algebra $(\mathcal{R}, *)$. Then $X \vartriangle Y = R(X) * Y + X * R(Y) - X * Y$.

Lemma 6.2. *For $n \in \mathbb{N}$ we have*

$$\ell(\alpha \prec P_n) = -\ell(\alpha)_{<0} * L^n \tag{6.11}$$

$$\ell(\alpha \succ P_n) = \ell(\alpha)_{\geqslant 0} * L^n \tag{6.12}$$

$$r(P_n \prec \alpha) = -L^n * r(\alpha)_{\geqslant 0} \tag{6.13}$$

$$r(P_n \succ \alpha) = L^n * r(\alpha)_{<0}. \tag{6.14}$$

Proof.

$$\begin{aligned}\ell(\alpha \prec P_{n+1}) &= \ell(\alpha \prec P_n \bullet P) = \ell((\alpha \prec P_n) \succ P) - \ell((\alpha \prec P_n) \prec P)\\ &= \ell(\alpha \prec P_n)_{\geqslant 0} * L + \ell(\alpha \prec P_n)_{<0} * L = \ell(\alpha \prec P_n) * L.\end{aligned}$$

Together with $\ell(\alpha \prec P) = -\ell(\alpha)_{<0} * L$, the first relation of the lemma follows by induction. The other relations are obtained in an analogous way. □

By iterated application of the preceding lemma, we obtain

$$L^{m_1,\ldots,m_k} = (-1)^{k-1}((\ldots(L^{m_1}{}_{<0} * L^{m_2})_{<0}\ldots)_{<0} * L^{m_{k-1}})_{<0} * L^{m_k} \tag{6.15}$$

$$\tilde{L}^{m_1,\ldots,m_k} := r\big(P_{m_1\ldots m_k}\big) = (-1)^{k-1} L^{m_1} * (L^{m_2} * (\ldots(L^{m_{k-1}} * L^{m_k}{}_{\geqslant 0})_{\geqslant 0}\ldots)_{\geqslant 0})_{\geqslant 0}. \tag{6.16}$$

Since the elements $P_{m_1\ldots m_k}$ defined in (4.4) span $\mathcal{A}(P)$, this allows us to compute $\ell(\alpha)$ and $r(\alpha)$ for all $\alpha \in \mathcal{A}(P)$.

Proposition 6.1. *In terms of*[26] $\overrightarrow{R}X := X_{\geqslant 0}$ *and* $X\overleftarrow{R} := X_{<0}$ *the following identity holds in* $\mathcal{R}$,

$$\begin{aligned}X_1 * \overrightarrow{R}X_2 * \cdots * \overrightarrow{R}X_k &= X_1\overleftarrow{R} * \cdots * X_{k-1}\overleftarrow{R} * X_k\\ &\quad + \sum_{j=1}^{k-1}(X_1\overleftarrow{R} * \cdots * X_{j-1}\overleftarrow{R} * X_j) \vartriangle (X_{j+1} * \overrightarrow{R}X_{j+2} * \cdots * \overrightarrow{R}X_k).\end{aligned} \tag{6.17}$$

Proof. The formula is easily verified for $k = 2$. The general formula is proved by induction on k. For $k+1$ we write the left-hand side as

$$X_1 * \overrightarrow{R}X_2 * \cdots * \overrightarrow{R}X_{k+1} = X_1 * \overrightarrow{R}X_2 * \cdots * \overrightarrow{R}X_{k-1} * \overrightarrow{R}(X_k * \overrightarrow{R}X_{k+1})$$

to which we can now apply the induction hypothesis. After use of the identities

$$X_k * \overrightarrow{R}X_{k+1} = (X_k\overleftarrow{R}) * X_{k+1} + X_k \vartriangle X_{k+1}$$

and

$$Y * (X_k \vartriangle X_{k+1}) - (Y * (\overrightarrow{R}X_k))\overleftarrow{R} * X_{k+1} - (Y * X_k) \vartriangle X_{k+1} = 0$$

for Y with $Y = Y\overleftarrow{R} = Y_{<0}$, the formula with k replaced by $k+1$ results. □

Corollary.

$$\tilde{L}^{m_1,\ldots,m_k} = L^{m_1,\ldots,m_k} - \sum_{j=1}^{k-1} L^{m_1,\ldots,m_j} \vartriangle \tilde{L}^{m_{j+1},\ldots,m_k} \tag{6.18}$$

[26] This notation avoids complex nested expressions such as those in (6.15) and (6.16). For example, $X_1 * \overrightarrow{R}X_2 * \cdots * \overrightarrow{R}X_k = X_1 * (X_2 * (\ldots(X_k)_{\geqslant 0})_{\geqslant 0}\ldots)_{\geqslant 0}$.

and thus

$$r(\alpha) = \ell(\alpha) - \sum \ell(\alpha_{(1)}) \vartriangle r(\alpha_{(2)}) \qquad \forall \alpha \in \mathcal{A}(P). \tag{6.19}$$

Lemma 6.3.

$$\begin{aligned} r\big(P_{m_1\dots m_k} \circ P_n\big) &= -L_{<0}^{m_1,\dots,m_k} * L^n - \sum_{j=1}^{k-1} L_{<0}^{m_1,\dots,m_j} * L^n * \tilde{L}_{<0}^{m_{j+1},\dots,m_k} \\ &\quad + L^n * \tilde{L}_{<0}^{m_1,\dots,m_k} - \sum_{j=1}^{k} L^{m_1,\dots,m_j} \vartriangle r\big(P_{m_{j+1}\dots m_k} \circ P_n\big). \end{aligned} \tag{6.20}$$

Proof. Using (2.13), the commutativity of $\circ$, and (2.7), we find

$$\begin{aligned} P_{m_1\dots m_k} \circ P_n &= \big(P_{m_1} \prec P_{m_2\dots m_k}\big) \circ P_n \\ &= \big(P_{m_1} \circ P_n - P_{m_1} \prec P_n\big) \prec P_{m_2\dots m_k} + P_{m_1} \prec \big(P_{m_2\dots m_k} \circ P_n\big) \\ &= P_n \succ P_{m_1\dots m_k} + P_{m_1} \prec \big(P_{m_2\dots m_k} \circ P_n\big) \end{aligned}$$

so that

$$\begin{aligned} r\big(P_{m_1\dots m_k} \circ P_n\big) &= r\big(P_{m_1} \prec P_{m_2\dots m_k} \circ P_n\big) + r\big(P_n \succ P_{m_1\dots m_k}\big) \\ &= -\ell\big(P_{m_1}\big) * r\big(P_{m_2\dots m_k} \circ P_n\big)_{\geqslant 0} + \ell(P_n) * r\big(P_{m_1\dots m_k}\big)_{<0} \\ &= -\ell\big(P_{m_1}\big)_{<0} * r\big(P_{m_2\dots m_k} \circ P_n\big) - \ell\big(P_{m_1}\big) \vartriangle r\big(P_{m_2\dots m_k} \circ P_n\big) \\ &\quad + \ell(P_n) * r\big(P_{m_1\dots m_k}\big)_{<0} \end{aligned}$$

and

$$\begin{aligned} \ell\big(P_{m_1}\big)_{<0} * r\big(P_{m_2\dots m_k} \circ P_n\big) &= \ell\big(P_{m_1}\big)_{<0} * \big(r\big(P_{m_2} \prec P_{m_3\dots m_k} \circ P_n\big) + r\big(P_n \succ P_{m_2\dots m_k}\big)\big) \\ &= \ell\big(P_{m_1}\big)_{<0} * \big(-L^{m_2} * r\big(P_{m_3\dots m_k} \circ P_n\big)_{\geqslant 0} + r\big(P_n \succ P_{m_2\dots m_k}\big)\big) \\ &= \ell\big(P_{m_1 m_2}\big) * r\big(P_{m_3\dots m_k} \circ P_n\big)_{\geqslant 0} + \ell\big(P_{m_1}\big)_{<0} * r\big(P_n \succ P_{m_2\dots m_k}\big) \\ &= \ell\big(P_{m_1 m_2}\big)_{<0} * r\big(P_{m_3\dots m_k} \circ P_n\big) + \ell\big(P_{m_1 m_2}\big) \vartriangle r\big(P_{m_3\dots m_k} \circ P_n\big) \\ &\quad + \ell\big(P_{m_1}\big)_{<0} * L^n * r\big(P_{m_2\dots m_k}\big)_{<0}. \end{aligned}$$

By iteration, we obtain

$$\begin{aligned} r\big(P_{m_1\dots m_k} \circ P_n\big) &= \ell(P_n) * r\big(P_{m_1\dots m_k}\big)_{<0} - \sum_{j=1}^{k-1} \ell\big(P_{m_1\dots m_j}\big) \vartriangle r\big(P_{m_{j+1}\dots m_k} \circ P_n\big) \\ &\quad - \sum_{j=1}^{k-2} \ell\big(P_{m_1\dots m_j}\big)_{<0} * L^n * r\big(P_{m_{j+1}\dots m_k}\big)_{<0} - \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * r\big(P_{m_k} \circ P_n\big). \end{aligned}$$

Next we convert the last term:

$$\begin{aligned} \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * r\big(P_{m_k} \circ P_n\big) &= \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * r\big(P_{m_k} \prec P_n + P_n \succ P_{m_k}\big) \\ &= \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * \big(-L^{m_k} * L^n{}_{\geqslant 0} + L^n * r\big(P_{m_k}\big)_{<0}\big) \\ &= \ell\big(P_{m_1\dots m_k}\big) * L^n{}_{\geqslant 0} + \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * L^n * r\big(P_{m_k}\big)_{<0} \\ &= \ell\big(P_{m_1\dots m_k}\big)_{<0} * L^n + \ell\big(P_{m_1\dots m_k}\big) \vartriangle r(P_n) + \ell\big(P_{m_1\dots m_{k-1}}\big)_{<0} * L^n * r\big(P_{m_k}\big)_{<0}. \end{aligned}$$

Insertion of this result into the previous formula yields (6.20). □

Lemma 6.4.

$$r\big(P_{m_1\ldots m_k}\circ(P_n\prec\alpha)\big) = -L^n * r\big(P_{m_1\ldots m_k}\circ\alpha\big)_{\geqslant 0} - \sum_{j=1}^{k} L^{m_1,\ldots,m_j}_{\geqslant 0} * L^n * r\big(P_{m_{j+1}\ldots m_k}\circ\alpha\big)_{\geqslant 0}$$
$$- \sum_{j=1}^{k} L^{m_1,\ldots,m_j} \vartriangle r\big(P_{m_{j+1}\ldots m_k}\circ(P_n\prec\alpha)\big). \tag{6.21}$$

Proof. First we note that (2.20) implies the identity

$$\begin{aligned} P_{m_1\ldots m_k}\circ(P_n\prec\alpha) &= \big(P_{m_1}\prec P_{m_2\ldots m_k}\big)\circ(P_n\prec\alpha) \\ &= P_{m_1}\prec\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big) + P_n\prec\big(P_{m_1\ldots m_k}\circ\alpha\big) + P_{m_1+n}\prec\big(P_{m_2\ldots m_k}\circ\alpha\big) \end{aligned}$$

so that

$$\begin{aligned} r\big(P_{m_1\ldots m_k}\circ(P_n\prec\alpha)\big) &= -L^{m_1} * r\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big)_{\geqslant 0} - L^n * r\big(P_{m_1\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad - L^{m_1+n} * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &= -L^{m_1}_{<0} * r\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big) - L^{m_1}\vartriangle r\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big) \\ &\quad - L^n * r\big(P_{m_1\ldots m_k}\circ\alpha\big)_{\geqslant 0} - L^{m_1+n} * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0}. \end{aligned}$$

This is a recursion formula, so we can rewrite the first term on the right-hand side as follows:

$$\begin{aligned} &L^{m_1}_{<0} * r\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big) \\ &\quad = L^{m_1}_{<0} * \big(-L^{m_2} * r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big)_{\geqslant 0} - L^n * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad\quad - L^{m_2+n} * r\big(P_{m_3\ldots m_k}\circ\alpha\big)_{\geqslant 0}\big) \\ &\quad = L^{m_1,m_2} * r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big)_{\geqslant 0} - L^{m_1}_{<0} * L^n * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad\quad + L^{m_1,m_2} * L^n * r\big(P_{m_3\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad = L^{m_1,m_2}_{<0} * r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big) + L^{m_1,m_2}\vartriangle r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big) \\ &\quad\quad - L^{m_1}_{<0} * L^n * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0} + L^{m_1,m_2} * L^n * r\big(P_{m_3\ldots m_k}\circ\alpha\big)_{\geqslant 0}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} r\big(P_{m_1\ldots m_k}\circ(P_n\prec\alpha)\big) &= -L^{m_1,m_2}_{<0} * r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big) - L^n * r\big(P_{m_1\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad - L^{m_1}_{\geqslant 0} * L^n * r\big(P_{m_2\ldots m_k}\circ\alpha\big)_{\geqslant 0} - L^{m_1,m_2} * L^n * r\big(P_{m_3\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad - L^{m_1}\vartriangle r\big(P_{m_2\ldots m_k}\circ(P_n\prec\alpha)\big) - L^{m_1,m_2}\vartriangle r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big). \end{aligned}$$

In the next step we proceed as follows:

$$\begin{aligned} L^{m_1,m_2}_{<0} * r\big(P_{m_3\ldots m_k}\circ(P_n\prec\alpha)\big) &= L^{m_1,m_2,m_3}_{<0} * r\big(P_{m_4\ldots m_k}\circ(P_n\prec\alpha)\big) \\ &\quad + L^{m_1,m_2,m_3}\vartriangle r\big(P_{m_4\ldots m_k}\circ(P_n\prec\alpha)\big) - L^{m_1,m_2}_{<0} * L^n * r\big(P_{m_3\ldots m_k}\circ\alpha\big)_{\geqslant 0} \\ &\quad + L^{m_1,m_2,m_3} * L^n * r\big(P_{m_4\ldots m_k}\circ\alpha\big)_{\geqslant 0} \end{aligned}$$

and so forth. In the last step, we have to use

$$P_{m_k}\circ(P_n\prec\alpha) = P_{m_k}\prec P_n\prec\alpha + P_n\prec\big(P_{m_k}\circ\alpha\big) + P_{m_k+n}\prec\alpha$$

which follows from (2.12) and (2.7), so that

$$\begin{aligned} L^{m_1,\ldots,m_{k-1}}_{<0} * r\big(P_{m_k}\circ(P_n\prec\alpha)\big) &= L^{m_1,\ldots,m_k}_{\geqslant 0} * L^n * r(\alpha)_{\geqslant 0} + L^{m_1,\ldots,m_k}\vartriangle r(P_n\prec\alpha) \\ &\quad - L^{m_1,\ldots,m_{k-1}} * L^n * r\big(P_{m_k}\circ\alpha\big)_{\geqslant 0}. \end{aligned}$$

Finally we obtain (6.21). □

6.2. Properties of the generalized derivations

Lemma 6.5.

$$\delta_{m_1\ldots m_k} L^n = -\left[L_{<0}^{m_1,\ldots,m_k}, L^n\right] + \sum_{j=1}^{k-1} \left(\delta_{m_1\ldots m_j} L^n\right) * L_{<0}^{m_{j+1},\ldots,m_k} \tag{6.22}$$

$$\delta_{m_1\ldots m_k} L^n = \left[L_{\geqslant 0}^{m_1,\ldots,m_k}, L^n\right] - \sum_{j=1}^{k-1} \left(\delta_{m_1\ldots m_j} L^n\right) * L_{\geqslant 0}^{m_{j+1},\ldots,m_k}. \tag{6.23}$$

Proof. By definition, the first equality holds for $n = 1$ and arbitrary $k \in \mathbb{N}$. Fix k and n and suppose it holds with k replaced by any $j \in \mathbb{N}$ with $j < k$ and n replaced by any $m \in \mathbb{N}$ with $m \leqslant n$. Using this and the generalized derivation property, we find

$$\begin{aligned}
\delta_{m_1\ldots m_k} L^{n+1} &= \left(\delta_{m_1\ldots m_k} L^n\right) * L + L^n * \delta_{m_1\ldots m_k} L + \sum_{j=1}^{k-1} \left(\delta_{m_1\ldots m_j} L^n\right) * \delta_{m_{j+1}\ldots m_k} L \\
&= -\left[L_{<0}^{m_1,\ldots,m_k}, L^{n+1}\right] + \sum_{i=1}^{k-1} \left(\delta_{m_1\ldots m_i} L^n\right) * \sum_{j=i}^{k-1} \left(\delta_{m_{i+1}\ldots m_j} L\right) * L_{<0}^{m_{j+1},\ldots,m_k} \\
&\quad + \sum_{j=1}^{k-1} L^n * \left(\delta_{m_1\ldots m_j} L\right) * L_{<0}^{m_{j+1},\ldots,m_k} \\
&= -\left[L_{<0}^{m_1,\ldots,m_k}, L^{n+1}\right] + \sum_{j=1}^{k-1} \left(\sum_{i=1}^{j} \left(\delta_{m_1\ldots m_i} L^n\right) * \delta_{m_{i+1}\ldots m_j} L \right. \\
&\quad \left. + L^n * \delta_{m_1\ldots m_j} L \right) * L_{<0}^{m_{j+1},\ldots,m_k} \\
&= -\left[L_{<0}^{m_1,\ldots,m_k}, L^{n+1}\right] + \sum_{j=1}^{k-1} \left(\delta_{m_1\ldots m_j} L^{n+1}\right) * L_{<0}^{m_{j+1},\ldots,m_k}
\end{aligned}$$

so that (6.22) also holds for $n+1$. Our second expression for $\delta_{m_1\ldots m_k} L^n$ now follows with the help of

$$\begin{aligned}
\left[L_{\geqslant 0}^{m_1,\ldots,m_k}, L^n\right] &= \left[L^{m_1,\ldots,m_k} - L_{<0}^{m_1,\ldots,m_k}, L^n\right] = -\left[L_{<0}^{m_1,\ldots,m_{k-1}} * L^{m_k}, L^n\right] - \left[L_{<0}^{m_1,\ldots,m_k}, L^n\right] \\
&= -\left[L_{<0}^{m_1,\ldots,m_{k-1}}, L^n\right] * L^{m_k} - \left[L_{<0}^{m_1,\ldots,m_k}, L^n\right] \\
&= \left(\delta_{m_1\ldots m_{k-1}} L^n - \sum_{j=1}^{k-2} \left(\delta_{m_1\ldots m_j} L^n\right) * L_{<0}^{m_{j+1},\ldots,m_{k-1}} \right) * L^{m_k} - \left[L_{<0}^{m_1,\ldots,m_k}, L^n\right] \\
&= \sum_{j=1}^{k-1} \left(\delta_{m_1\ldots m_j} L^n\right) * L^{m_{j+1},\ldots,m_k} - \left[L_{<0}^{m_1,\ldots,m_k}, L^n\right].
\end{aligned}$$

□

In particular, we have

$$\delta_m(\partial) = \delta_m(L_{\geqslant 0}) = (\delta_m L)_{\geqslant 0} = (-[(L^m)_{<0}, L])_{\geqslant 0} = 0. \tag{6.24}$$

By induction, using (6.22), it follows that

$$\delta_{m_1\ldots m_k}(\partial) = 0 \qquad k = 1, 2, \ldots. \tag{6.25}$$

Using the fact that ∂^{-1} is the inverse of ∂, this in turn implies

$$\delta_{m_1\ldots m_k}(\partial^{-1}) = 0 \qquad k = 1, 2, \ldots. \tag{6.26}$$

The main result of this subsection is stated next.

Proposition 6.2.

$$\delta_{m_1\ldots m_k}\ell(\alpha) = \ell\big(P_{m_1\ldots m_k} \circ \alpha\big) - \sum_{j=0}^{k-1} \ell\big(P_{m_1\ldots m_j} \circ \alpha\big) \vartriangle r\big(P_{m_{j+1}\ldots m_k}\big) \tag{6.27}$$

$$\delta_{m_1\ldots m_k}r(\alpha) = r\big(P_{m_1\ldots m_k} \circ \alpha\big) + \sum_{j=1}^{k} \ell\big(P_{m_1\ldots m_j}\big) \vartriangle r\big(P_{m_{j+1}\ldots m_k} \circ \alpha\big). \tag{6.28}$$

The remainder of this subsection is devoted to the proof of this proposition. Let us define

$$\delta'_{m_1\ldots m_k}\ell(\alpha) := \ell\big(P_{m_1\ldots m_k} \circ \alpha\big) - \sum_{j=0}^{k-1} \ell\big(P_{m_1\ldots m_j} \circ \alpha\big) \vartriangle r\big(P_{m_{j+1}\ldots m_k}\big) \tag{6.29}$$

$$\delta''_{m_1\ldots m_k}r(\alpha) := r\big(P_{m_1\ldots m_k} \circ \alpha\big) + \sum_{j=1}^{k} \ell\big(P_{m_1\ldots m_j}\big) \vartriangle r\big(P_{m_{j+1}\ldots m_k} \circ \alpha\big). \tag{6.30}$$

We have to show that $\delta'_{m_1\ldots m_k}$ and $\delta''_{m_1\ldots m_k}$ coincide with $\delta_{m_1\ldots m_k}$ on $\ell(\mathcal{A}(P))$, respectively $r(\mathcal{A}(P))$.

Lemma 6.6.

$$\delta'_{m_1\ldots m_k}L^n = \delta''_{m_1\ldots m_k}L^n = \delta_{m_1\ldots m_k}L^n. \tag{6.31}$$

Proof. First we note that

$$\delta''_{m_1\ldots m_k}L^n = \delta''_{m_1\ldots m_k}r(P_n) = r\big(P_{m_1\ldots m_k} \circ P_n\big) + \sum_{j=1}^{k} \ell\big(P_{m_1\ldots m_j}\big) \vartriangle r\big(P_{m_{j+1}\ldots m_k} \circ P_n\big)$$

which can be further evaluated with the help of (6.20),

$$\delta''_{m_1\ldots m_k}L^n = -L^{m_1,\ldots,m_k}_{<0} * L^n - \sum_{j=1}^{k-1} L^{m_1,\ldots,m_j}_{<0} * L^n * \tilde{L}^{m_{j+1},\ldots,m_k}_{<0} + L^n * \tilde{L}^{m_1,\ldots,m_k}_{<0}.$$

Next we use (6.18) and $(X \vartriangle Y)_{<0} = -X_{<0} * Y_{<0}$ to obtain

$$\delta''_{m_1\ldots m_k}L^n + \big[L^{m_1,\ldots,m_k}_{<0}, L^n\big] = -\sum_{j=1}^{k-1} \big[L^{m_1,\ldots,m_j}_{<0}, L^n\big] * \tilde{L}^{m_{j+1},\ldots,m_k}_{<0}.$$

Using this formula, we will prove by induction that $\delta''_{m_1\ldots m_k}L^n$ equals the right-hand side of (6.22). For $k = 2$, the last relation reads

$$\delta''_{m_1m_2}L^n = -\big[L^{m_1,m_2}_{<0}, L^n\big] - \big[L^{m_1}_{<0}, L^n\big] * \tilde{L}^{m_2}_{<0} = -\big[L^{m_1,m_2}_{<0}, L^n\big] + \big(\delta_{m_1}L^n\big) * L^{m_2}_{<0}$$

where we used $\delta_m L^n = -\big[L^m_{<0}, L^n\big]$. Let us now fix k and assume that $\delta''_{m_1\dots m_k} L^n = \delta_{m_1\dots m_k} L^n$ holds for $m_1, \dots, m_j$ with $2 \leqslant j \leqslant k$. Then it also holds for $k+1$ since

$$\begin{aligned}
\delta''_{m_1\dots m_{k+1}} L^n + \big[L^{m_1,\dots,m_{k+1}}_{<0}, L^n\big] &= -\sum_{j=1}^{k} \big[L^{m_1,\dots,m_j}_{<0}, L^n\big] * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} \\
&= \sum_{j=1}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} - \sum_{j=1}^{k}\sum_{l=1}^{j-1} \big(\delta_{m_1\dots m_l} L^n\big) * L^{m_{l+1},\dots,m_j}_{<0} * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} \\
&= \sum_{j=1}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} - \sum_{l=1}^{k-1} \big(\delta_{m_1\dots m_l} L^n\big) * \sum_{j=l+1}^{k} L^{m_{l+1},\dots,m_j}_{<0} * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} \\
&= \sum_{j=1}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * L^{m_{j+1},\dots,m_{k+1}}_{<0}
\end{aligned}$$

using again the 'negative' part of (6.18) in the form

$$\sum_{j=l+1}^{k} L^{m_1,\dots,m_j}_{<0} * \tilde{L}^{m_{j+1},\dots,m_{k+1}}_{<0} = \tilde{L}^{m_{l+1},\dots,m_{k+1}}_{<0} - L^{m_{l+1},\dots,m_{k+1}}_{<0}.$$

The equality $\delta'_{m_1\dots m_k} = \delta_{m_1\dots m_k}$ is obtained in the same way. □

Lemma 6.7. *The following are identities for all $n \in \mathbb{N}$,*

$$\delta'_{m_1\dots m_k}(\ell(\alpha)_{<0} * L^n) = \sum_{j=0}^{k} \big(\delta'_{m_1\dots m_j}\ell(\alpha)\big)_{<0} * \delta_{m_{j+1}\dots m_k} L^n \tag{6.32}$$

$$\delta''_{m_1\dots m_k}(L^n * r(\alpha)_{\geqslant 0}) = \sum_{j=0}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * \big(\delta''_{m_{j+1}\dots m_k} r(\alpha)\big)_{\geqslant 0}. \tag{6.33}$$

Proof. Using (6.21), we obtain

$$\delta''_{m_1\dots m_k} r(P_n \prec \alpha) = -L^n * r\big(P_{m_1\dots m_k} \circ \alpha\big)_{\geqslant 0} - \sum_{j=1}^{k} L^{m_1,\dots,m_j}_{\geqslant 0} * L^n * r\big(P_{m_{j+1}\dots m_k} \circ \alpha\big)_{\geqslant 0}$$

and thus

$$\delta''_{m_1\dots m_k} r(P_n \prec \alpha) = -L^n * \delta''_{m_1\dots m_k} r(\alpha)_{\geqslant 0} - \sum_{j=1}^{k} \big[L^{m_1,\dots,m_j}_{\geqslant 0}, L^n\big] * r\big(P_{m_{j+1}\dots m_k} \circ \alpha\big)_{\geqslant 0}.$$

Now we eliminate the commutator via (6.23) to get

$$\begin{aligned}
\delta''_{m_1\dots m_k} r(P_n \prec \alpha) &= -L^n * \delta''_{m_1\dots m_k} r(\alpha)_{\geqslant 0} - \sum_{j=1}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * r\big(P_{m_{j+1}\dots m_k} \circ \alpha\big)_{\geqslant 0} \\
&\quad + \sum_{j=2}^{k}\sum_{l=1}^{j-1} \big(\delta_{m_1\dots m_l} L^n\big) * L^{m_{l+1},\dots,m_j}_{\geqslant 0} * r\big(P_{m_{j+1}\dots m_k} \circ \alpha\big)_{\geqslant 0} \\
&= -L^n * \delta''_{m_1\dots m_k} r(\alpha)_{\geqslant 0} - \sum_{j=1}^{k} \big(\delta_{m_1\dots m_j} L^n\big) * r\big(P_{m_{j+1}\dots m_k} \circ \alpha\big)_{\geqslant 0}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-1}\left(\delta_{m_1\ldots m_l}L^n\right)*\sum_{j=l+1}^{k}\ell\left(P_{m_{l+1}\ldots m_j}\right)_{\geqslant 0}*r\left(P_{m_{j+1}\ldots m_k}\circ\alpha\right)_{\geqslant 0}\\
&= -L^n*\delta''_{m_1\ldots m_k}r(\alpha)_{\geqslant 0}-\left(\delta_{m_1\ldots m_k}L^n\right)*r(\alpha)_{\geqslant 0}\\
&\quad -\sum_{l=1}^{k-1}\left(\delta_{m_1\ldots m_l}L^n\right)*\left(r\left(P_{m_{l+1}\ldots m_k}\circ\alpha\right)\right.\\
&\quad \left.+\sum_{j=l+1}^{k}\ell\left(P_{m_{l+1}\ldots m_j}\right)\vartriangle r\left(P_{m_{j+1}\ldots m_k}\circ\alpha\right)\right)_{\geqslant 0}\\
&= -L^n*\delta''_{m_1\ldots m_k}r(\alpha)_{\geqslant 0}-\left(\delta_{m_1\ldots m_k}L^n\right)*r(\alpha)_{\geqslant 0}\\
&\quad -\sum_{l=1}^{k-1}\left(\delta_{m_1\ldots m_l}L^n\right)*\left(\delta''_{m_{l+1}\ldots m_k}r(\alpha)\right)_{\geqslant 0}\\
&= -\sum_{l=0}^{k}\left(\delta_{m_1\ldots m_l}L^n\right)*\left(\delta''_{m_{l+1}\ldots m_k}r(\alpha)\right)_{\geqslant 0}.
\end{aligned}
$$

The proof of (6.33) is completed by inserting $r(P_n \prec \alpha) = -L^n * r(\alpha)_{\geqslant 0}$. The other identity can be proved in a similar way. □

For the moment, let us simply write δ' instead of $\delta'_{m_1\ldots m_k}$. By iterative use of (6.32), respectively (6.33), we find

$$
\begin{aligned}
\delta'\ell\left(P_{n_1\ldots n_j}\right) &= (-1)^{j-1}\sum(\delta'_{(1)}L^{n_1})\overleftarrow{R}*\cdots*(\delta'_{(j-1)}L^{n_{j-1}})\overleftarrow{R}*(\delta'_{(j)}L^{n_j})\\
\delta''r\left(P_{n_1\ldots n_j}\right) &= (-1)^{j-1}\sum(\delta_{(1)}L^{n_1})*\overrightarrow{R}(\delta''_{(2)}L^{n_2})*\cdots*\overrightarrow{R}(\delta''_{(j)}L^{n_j})
\end{aligned}
$$

using the projection operators defined in proposition 6.1 and a Sweedler notation. According to lemma 6.6 we may drop the primes on the right-hand sides of these equations. Using the generalized derivation property of $\delta_{m_1\ldots m_k}$, we obtain

$$
\delta'\ell\left(P_{n_1\ldots n_j}\right)=\delta\ell\left(P_{n_1\ldots n_j}\right) \qquad \delta''r\left(P_{n_1\ldots n_j}\right)=\delta r\left(P_{n_1\ldots n_j}\right).
$$

Since the elements $P_{n_1\ldots n_j}$ span $\mathcal{A}(P)$, this proves proposition 6.2.

6.3. Commutativity of the generalized derivations

Lemma 6.8.

$$
\ell\left(P_{m_1\ldots m_k}\circ\alpha\right)=\delta_{m_1\ldots m_k}\ell(\alpha)+\sum_{j=0}^{k-1}\delta_{m_1\ldots m_j}\ell(\alpha)\vartriangle\ell\left(P_{m_{j+1}\ldots m_k}\right). \tag{6.34}
$$

Proof. By induction. For $k=1$ this follows directly from (6.27) using $r(P_m)=\ell(P_m)$. Let us fix k and assume that (6.34) holds for $1\leqslant k'\leqslant k$. Starting with (6.27), we obtain

$$
\begin{aligned}
\ell\left(P_{m_1\ldots m_{k+1}}\circ\alpha\right)-\delta_{m_1\ldots m_{k+1}}\ell(\alpha) &= \sum_{j=0}^{k}\ell\left(P_{m_1\ldots m_j}\circ\alpha\right)\vartriangle r\left(P_{m_{j+1}\ldots m_{k+1}}\right)\\
&= \sum_{j=0}^{k}\delta_{m_1\ldots m_j}\ell(\alpha)\vartriangle r\left(P_{m_{j+1}\ldots m_{k+1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{k}\sum_{l=0}^{j-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\ell\big(P_{m_{l+1}\dots m_j}\big)\vartriangle r\big(P_{m_{j+1}\dots m_{k+1}}\big)\\
&=\sum_{j=0}^{k}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle r\big(P_{m_{j+1}\dots m_{k+1}}\big)\\
&\quad+\sum_{l=0}^{k-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\sum_{j=l+1}^{k}\ell\big(P_{m_{l+1}\dots m_j}\big)\vartriangle r\big(P_{m_{j+1}\dots m_{k+1}}\big)\\
&=\sum_{j=0}^{k}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle r\big(P_{m_{j+1}\dots m_{k+1}}\big)\\
&\quad+\sum_{l=0}^{k-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\Big(\ell\big(P_{m_{l+1}\dots m_{k+1}}\big)-r\big(P_{m_{l+1}\dots m_{k+1}}\big)\Big)\\
&=\delta_{m_1\dots m_k}\ell(\alpha)\vartriangle r\big(P_{m_{k+1}}\big)+\sum_{j=0}^{k-1}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle\ell\big(P_{m_{j+1}\dots m_{k+1}}\big)\\
&=\sum_{j=0}^{k}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle\ell\big(P_{m_{j+1}\dots m_{k+1}}\big).
\end{aligned}
$$

Hence (6.34) also holds for $k+1$. □

Proposition 6.3.

$$
\delta_{m_1\dots m_k}(\ell(\alpha)\vartriangle r(\beta))=\sum_{j=0}^{k}\ell\big(P_{m_1\dots m_j}\circ\alpha\big)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big).\tag{6.35}
$$

Proof. With the help of (6.34), we find

$$
\begin{aligned}
&\sum_{j=0}^{k}\ell\big(P_{m_1\dots m_j}\circ\alpha\big)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\\
&\quad=\sum_{j=0}^{k}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\\
&\quad\quad+\sum_{j=1}^{k}\sum_{l=0}^{j-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\ell\big(P_{m_{l+1}\dots m_j}\big)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\\
&\quad=\sum_{j=0}^{k}\delta_{m_1\dots m_j}\ell(\alpha)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\\
&\quad\quad+\sum_{l=0}^{k-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\sum_{j=l+1}^{k}\ell\big(P_{m_{l+1}\dots m_j}\big)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\\
&\quad=\sum_{l=0}^{k-1}\delta_{m_1\dots m_l}\ell(\alpha)\vartriangle\left(r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)+\sum_{j=l+1}^{k}\ell\big(P_{m_{l+1}\dots m_j}\big)\vartriangle r\big(P_{m_{j+1}\dots m_k}\circ\beta\big)\right)\\
&\quad\quad+\delta_{m_1\dots m_k}\ell(\alpha)\vartriangle r(\beta)
\end{aligned}
$$

$$= \sum_{l=0}^{k-1} \delta_{m_1\ldots m_l}\ell(\alpha) \vartriangle \delta_{m_{j+1}\ldots m_k} r(\beta) + \delta_{m_1\ldots m_k}\ell(\alpha) \vartriangle r(\beta)$$

$$= \sum_{l=0}^{k} \delta_{m_1\ldots m_l}\ell(\alpha) \vartriangle \delta_{m_{j+1}\ldots m_k} r(\beta).$$

This equals $\delta_{m_1\ldots m_k}(\ell(\alpha) \vartriangle r(\beta))$ by use of the generalized derivation rule (6.8). $\square$

Theorem 6.1.

$$[\delta_{m_1\ldots m_k}, \delta_{n_1\ldots n_l}] = 0. \tag{6.36}$$

Proof. Using (6.28), we obtain

$$\delta_{m_1\ldots m_k}\delta_{n_1\ldots n_l} r(\alpha) = r(P_{m_1\ldots m_k} \circ P_{n_1\ldots n_l} \circ \alpha) + \sum_{j=1}^{k} \ell(P_{m_1\ldots m_j}) \vartriangle r(P_{m_{j+1}\ldots m_k} \circ P_{n_1\ldots n_l} \circ \alpha)$$
$$+ \sum_{j=1}^{l} \delta_{m_1\ldots m_k}(\ell(P_{n_1\ldots n_j}) \vartriangle r(P_{n_{j+1}\ldots n_l} \circ \alpha))$$

where, according to (6.35),

$$\delta_{m_1\ldots m_k}(\ell(P_{n_1\ldots n_j}) \vartriangle r(P_{n_{j+1}\ldots n_l} \circ \alpha))$$
$$= \sum_{p=0}^{k} \ell(P_{m_1\ldots m_p} \circ P_{n_1\ldots n_j}) \vartriangle r(P_{m_{p+1}\ldots m_k} \circ P_{n_{j+1}\ldots n_l} \circ \alpha)$$
$$= \sum_{p=1}^{k} \ell(P_{m_1\ldots m_p} \circ P_{n_1\ldots n_j}) \vartriangle r(P_{m_{p+1}\ldots m_k} \circ P_{n_{j+1}\ldots n_l} \circ \alpha)$$
$$+ \ell(P_{n_1\ldots n_j}) \vartriangle r(P_{m_1\ldots m_k} \circ P_{n_{j+1}\ldots n_l} \circ \alpha).$$

The commutativity of the $\circ$ product now implies that $[\delta_{m_1\ldots m_k}, \delta_{n_1\ldots n_l}] = 0$ on $r(\mathcal{A}(P))$. A similar argument shows that this also holds on $\ell(\mathcal{A}(P))$. The generalized derivation property (6.8) extends this commutation relation to the algebra generated by $\ell(\mathcal{A}(P)) \cup r(\mathcal{A}(P))$ and ∂^{-1}, taking (6.26) into account (and using the product $*$ and the projection $()_{\geqslant 0}$). But this reaches the whole of $\mathcal{R}$. $\square$

6.4. Taking the residue

In this subsection we explore the properties of the map Φ defined in (6.9). According to (6.18) and (6.2) we also have $\Phi(\alpha) = \mathrm{res}(r(\alpha))$. An immediate consequence of (6.10) is

$$\Phi(P_n) = \mathrm{res}(L^n) \tag{6.37}$$

and from definition (6.5) we get

$$\Phi(P_{m_1\ldots m_k}) = \mathrm{res}(L^{m_1,\ldots,m_k}). \tag{6.38}$$

Proposition 6.4.

$$\Phi(\alpha \prec \beta) = -\mathrm{res}(\ell(\alpha) * r(\beta)_{\geqslant 0}) \tag{6.39}$$

$$\Phi(\alpha \succ \beta) = \mathrm{res}(\ell(\alpha) * r(\beta)_{<0}). \tag{6.40}$$

Proof. For $\beta \in \mathcal{A}^1(P)$, it is sufficient to consider

$$\mathrm{res}(\ell(\alpha \prec P_n)) = -\mathrm{res}(\ell(\alpha)_{<0} * r(P_n)) = -\mathrm{res}(\ell(\alpha) * r(P_n)_{\geqslant 0})$$

by use of (6.10) and (6.11). Let us assume that (6.39) holds for $\beta \in \mathcal{A}(P)$ of degree $\leqslant n$, and for all $\alpha \in \mathcal{A}(P)$. Then (6.39) also holds for $\beta \in \mathcal{A}(P)$ of degree $n+1$ since

$$\begin{aligned}\mathrm{res}(\ell(\alpha \prec (P_m \prec \beta))) &= \mathrm{res}(\ell((\alpha \prec P_m) \prec \beta)) = -\mathrm{res}(\ell(\alpha \prec P_m) * r(\beta)_{\geqslant 0}) \\ &= \mathrm{res}(\ell(\alpha)_{<0} * L^m * r(\beta)_{\geqslant 0}) = -\mathrm{res}(\ell(\alpha)_{<0} * r(P_m \prec \beta)) \\ &= -\mathrm{res}(\ell(\alpha) * r(P_m \prec \beta)_{\geqslant 0}).\end{aligned}$$

The proof of the second relation proceeds in the same way. □

Theorem 6.2. *Φ has the following homomorphism property:*

$$\Phi(\alpha \,\hat{\times}\, \beta) = \Phi(\alpha) * \Phi(\beta). \tag{6.41}$$

Proof.

$$\begin{aligned}\Phi(\alpha \,\hat{\times}\, \beta) &= -\Phi(\alpha \prec P \succ \beta) = \mathrm{res}(\ell(\alpha)_{<0} * L * r(\beta)_{<0}) \\ &= \mathrm{res}(\ell(\alpha)_{<0} * \partial r(\beta)_{<0}) = \mathrm{res}(\ell(\alpha)) * \mathrm{res}(r(\beta)).\end{aligned}$$ □

Lemma 6.9.

$$\delta_{m_1 \ldots m_k}\, \mathrm{res}\, X = \mathrm{res}\, \delta_{m_1 \ldots m_k} X \qquad \forall X \in \mathcal{R}. \tag{6.42}$$

Proof. Using the identity $\mathrm{res}\, X = (X_{<0}\partial)_{\geqslant 0}$ and writing simply δ for $\delta_{m_1 \ldots m_k}$, we have

$$\delta\, \mathrm{res}\, X = \delta(X_{<0}\partial)_{\geqslant 0} = (\delta(X_{<0}\partial))_{\geqslant 0} = ((\delta X_{<0})\partial)_{\geqslant 0} = ((\delta X)_{<0}\partial)_{\geqslant 0} = \mathrm{res}\, \delta X$$

where we used (6.7), (6.8) and (6.25). □

Proposition 6.5.

$$\delta_{m_1 \ldots m_k} \Phi(\alpha) = \Phi\big(P_{m_1 \ldots m_k} \circ \alpha\big). \tag{6.43}$$

Proof. Taking the residue of (6.28) and using (6.2), leads to

$$\delta_{m_1 \ldots m_k} \Phi(\alpha) = \mathrm{res}\big(\delta_{m_1 \ldots m_k} r(\alpha)\big) = \mathrm{res}\, r\big(P_{m_1 \ldots m_k} \circ \alpha\big) = \Phi\big(P_{m_1 \ldots m_k} \circ \alpha\big)$$ □

7. Back to the (x)ncKP hierarchy

The formalism developed in the preceding section will now be applied to recover properties of the ncKP and xncKP hierarchies from certain sets of algebraic identities in $\mathcal{A}(P)$.

7.1. The ncKP hierarchy

Since according to theorem 6.1 the δ_n, $n \in \mathbb{N}$, are commuting derivations, we may set $\delta_n = \partial_{t_n}$ on $\mathcal{R}$. The equations

$$L_{t_n} = \delta_n L \qquad n = 1, 2, \ldots \tag{7.1}$$

are then compatible. These are the defining relations (1.3) of the ncKP hierarchy. An immediate consequence is

$$\Phi(P_n) = \partial_{t_n} \phi \tag{7.2}$$

which is (1.5). Furthermore, proposition 6.5 leads to $\Phi(P_n \circ \alpha) = \partial_{t_n}\Phi(\alpha)$. In the following, the fundamental homomorphism property $\Phi(\alpha \hat{\times} \beta) = \Phi(\alpha) * \Phi(\beta)$ (theorem 6.2) will also play an important role. Applying Φ, for example, to the identity (4.24), results in the ncKP equation (1.6).

Let us recall the definitions

$$(L^n)_{<0} = -\sum_{m=1}^{\infty} \sigma_m^{(n)} * L^{-m} = \sum_{m=1}^{\infty} L^{-m} * \eta_m^{(n)} \tag{7.3}$$

of coefficients $\sigma_m^{(n)}$ and $\eta_m^{(n)}$ from [6], where iteration formulae for the (x)ncKP hierarchy equations were derived in terms of them. The σ-coefficients frequently appeared in treatments of the 'commutative' KP hierarchy (see [43], for example).

Theorem 7.1.

$$\Phi(U_n) = u_n \qquad \Phi\big(C_n^{(m)}\big) = \sigma_n^{(m)} \qquad \Phi\big(H_n^{(m)}\big) = \eta_n^{(m)} \tag{7.4}$$

with U_n, $C_n^{(m)}$, $H_n^{(m)}$ defined in section 4.1.

Proof. Using (6.28), (6.1) and $\delta_1 = [\partial, \cdot]$, we obtain

$$r(P \circ \alpha)_{\geqslant 0} = (\delta_1 r(\alpha) - L \vartriangle r(\alpha))_{\geqslant 0} = (\partial r(\alpha) - r(\alpha)\partial - \partial r(\alpha)_{\geqslant 0})_{\geqslant 0}.$$

Since $[\partial, X_{<0}]_{\geqslant 0} = 0$ for all $X \in \mathcal{R}$, this implies $r(P \circ \alpha)_{\geqslant 0} = -r(\alpha)_{\geqslant 0}\partial$ which can be applied iteratively to the expression

$$r(U_n) = (-1)^n r(P \prec P^{\circ(n-2)}) = -(-1)^n L * r(P \circ P^{\circ(n-3)})_{\geqslant 0}$$

to yield $\Phi(U_n) = \mathrm{res}(r(U_n)) = \mathrm{res}(L\partial^{n-2}) = u_n$.

With the help of (6.39) and (6.4), the second relation of the theorem is obtained as follows,

$$\begin{aligned}\Phi\big(C_n^{(m)}\big) &= (-1)^n\,\mathrm{res}(\ell(P_m \prec P^{\prec n-1})) = (-1)^{n+1}\,\mathrm{res}(\ell(P_m)_{<0} * r(P^{\prec n-1}))\\ &= (-1)^n \sum_{k=1}^{\infty} \sigma_k^{(m)} * \mathrm{res}(L^{-k} * r(P^{\prec n-1}))\\ &= (-1)^{n+1} \sum_{k=1}^{\infty} \sigma_k^{(m)} * \mathrm{res}(L^{-k+1} * r(P^{\prec n-2})_{\geqslant 0})\\ &= (-1)^{n+1} \sum_{k=1}^{\infty} \sigma_k^{(m)} * \mathrm{res}((L^{-k+1})_{<0} * r(P^{\prec n-2}))\\ &= (-1)^{n+1} \sum_{k=2}^{\infty} \sigma_k^{(m)} * \mathrm{res}(L^{-k+1} * r(P^{\prec n-2})) = \cdots\\ &= \sum_{k=n-1}^{\infty} \sigma_k^{(m)} * \mathrm{res}(L^{n-2-k} * r(P)) = \sum_{k=n}^{\infty} \sigma_k^{(m)} * \mathrm{res}(L^{n-1-k}) = \sigma_n^{(m)}\end{aligned}$$

since $\mathrm{res}(L^{-l}) = 1$ if $l = 1$ and $\mathrm{res}(L^{-l}) = 0$ if $l > 1$. The last relation of the theorem is verified in a similar way (see also the proof of theorem 8.1). $\square$

By application of the above results, making use of theorem 6.2 and proposition 6.5, Φ maps the identity (4.23) to

$$\partial_{t_m}\partial_{t_n}\phi = \sigma_{m+1}^{(n)} + \eta_{m+1}^{(n)} + \sum_{r=1}^{m-1} \big(\sigma_{m-r}^{(n)} * \partial_{t_r}\phi + \partial_{t_r}\phi * \eta_{m-r}^{(n)}\big) \tag{7.5}$$

which is (5.31) in [6]. Via (the image under Φ of) algebraic relations obtained in section 4.1 this equation determines iteratively a 'complete' set of ncKP hierarchy equations in the sense that any equation for ϕ which arises from the hierarchy can be expressed as a combination of such equations[27]. Hence, the ncKP hierarchy lies in the image of a set of identities in $\mathcal{A}(P)$ under the map Φ. According to results in section 4.1, we know that the respective set of identities in $\mathcal{A}(P)$ can be built from $P_m, m \in \mathbb{N}$, solely by use of the products $\circ$ and $\hat{\times}$. We expect that also the following statement holds.

Conjecture. All *identities in $\mathcal{A}(P)$, which are built from $P_m, m \in \mathbb{N}$, only with the help of the products $\circ$ and $\hat{\times}$, are mapped by Φ to ncKP equations (expressed in terms of the potential ϕ).*

If there were such an identity which is *not* mapped by Φ to an ncKP equation, we know that it would be mapped to an interesting equation since the latter would be solvable via the ansatz described in the introduction and thus admit multiple 'soliton' solutions. We believe, however, that the ncKP hierarchy exhausts the corresponding possibilities (under the restrictions stated in the conjecture).

7.2. Extension of the Moyal-deformed ncKP hierarchy

According to (6.43),

$$\vartheta_{mn} := \tfrac{1}{2}(\delta_{mn} - \delta_{nm}) \tag{7.6}$$

satisfies

$$\Phi(A_{mn} \circ \alpha) = \vartheta_{mn} \Phi(\alpha) \tag{7.7}$$

with A_{mn} defined in (4.19), and (6.8) leads to

$$\vartheta_{mn}(X * Y) = (\vartheta_{mn} X) * Y + X * \vartheta_{mn} Y + \tfrac{1}{2}(\delta_m X * \delta_n Y - \delta_n X * \delta_m Y) \tag{7.8}$$

which allows us to set $\vartheta_{mn} = \partial_{\theta_{mn}}$ on $\mathcal{R}$ (where $\partial_{\theta_{mn}}$ is the partial derivative with respect to the deformation parameter θ_{mn} entering the Moyal $*$-product (1.16)), provided that also δ_n is set equal to ∂_{t_n} (which yields the ncKP hierarchy). Since, according to theorem 6.1, $\vartheta_{mn}, m, n \in \mathbb{N}$, commute with each other and also with $\delta_n, n \in \mathbb{N}$, the equations

$$L_{\theta_{mn}} = \vartheta_{mn} L \tag{7.9}$$

are compatible and extend the Moyal-deformed ncKP hierarchy[28]. In this way, one recovers the extension of the Moyal-deformed ncKP hierarchy obtained in [5] and further explored in [6].

From (6.23) we obtain

$$\delta_{mn} L^r = [(L^{m,n})_{\geqslant 0}, L^r] - (\delta_m L^r) * (L^n)_{\geqslant 0} \tag{7.10}$$

and therefore

$$\vartheta_{mn} L^r = [W^{(m,n)}, L^r]_* + \tfrac{1}{2}(\delta_n L^r * (L^m)_{\geqslant 0} - \delta_m L^r * (L^n)_{\geqslant 0}) \tag{7.11}$$

with

$$W^{(m,n)} := \tfrac{1}{2}(L^{m,n} - L^{n,m})_{\geqslant 0} = \tfrac{1}{2}((L^n)_{<0} * L^m - (L^m)_{<0} * L^n)_{\geqslant 0} \tag{7.12}$$

using $L^{m,n} = -(L^m)_{<0} * L^n$ in the last step.

[27] With the choice $m = 1$, after an x-integration the last equation can be solved for ϕ_{t_n} if $n > 2$ [6].

[28] The xncKP flow given by (7.9) for fixed m, n only commutes with the corresponding flow of the same equation with m, n replaced by another pair r, s of natural numbers, if the ncKP equations associated with the evolution parameters t_m, t_n, t_r, t_s are satisfied (see also [6]). The proof of theorem 6.1 clearly manifests this dependence of 'second-order' flows on those of 'first order'.

Replacing ϑ_{mn} by $\partial_{\theta_{mn}}$ in (7.11) for $r = 1$, taking the residue and performing an x-integration, leads to

$$\partial_{\theta_{mn}}\phi = \tfrac{1}{2}\operatorname{res}(L^{m,n} - L^{n,m}) = \Phi(A_{mn}). \tag{7.13}$$

By application of Φ to identities in $\mathcal{A}(P)$ involving besides P_m also A_{mn}, and otherwise built with the products $\circ$ and $\hat{\times}$ only, we obtain explicit xncKP equations beyond those of the ncKP hierarchy. In fact, applying Φ to (4.22), we reach all those equations, since we recover (5.30) in [6]. Recalling results of section 4.1, this proves that there is a set of identities in $\mathcal{A}(P)$, which can be expressed solely in terms of $P_m, A_{mn}, m, n \in \mathbb{N}$, and the products $\circ$ and $\hat{\times}$, such that Φ maps it to a complete set of xncKP equations for the potential ϕ. Probably *all* identities built in this way are mapped by Φ to xncKP equations.

Remark. It is well known (see [44], for example) that by means of an equivalence transformation

$$f *' g = D^{-1}((Df) * (Dg)) \tag{7.14}$$

with an invertible operator D one can eliminate a possible *symmetric* part of the deformation parameters θ_{mn} from the $*$-product. Let us see how the algebra $\mathcal{A}(P)$ reflects this fact. For the moment, let us generalize θ_{mn} to t_{mn} by adding a symmetric part. From the definition of the main product $\circ$, we have the identity

$$P_{mn} + P_{nm} = P_m \circ P_n - P_{m+n} \tag{7.15}$$

in $\mathcal{A}(P)$. This is mapped by Φ to the *linear* equation

$$\phi_{t_{mn}} + \phi_{t_{nm}} = \phi_{t_m t_n} - \phi_{t_{m+n}} \tag{7.16}$$

which is equivalent to

$$\phi_{t_{mn}} = \phi_{\theta_{mn}} + \tfrac{1}{2}\big(\phi_{t_m t_n} - \phi_{t_{m+n}}\big) \tag{7.17}$$

and allows us to express the partial derivative with respect to the symmetric part of t_{mn} in terms of partial derivatives with respect to the variables t_n. We may therefore restrict our considerations to the antisymmetric combination $A_{mn} = (P_{mn} - P_{nm})/2$ and thus the antisymmetric part θ_{mn} of t_{mn}.

8. Beyond Moyal deformation: XncKP hierarchy

In this section, we replace the Moyal product by an associative $*$-product which may be regarded as including all (at least in the present framework) possible deformations. This leads us to an extension of the ncKP hierarchy which is even bigger than the xncKP hierarchy.

8.1. Maximal deformation $*$-product

Now we allow the coefficients of L to depend on variables $t_{(r)} = \big\{t_{m_1\ldots m_r}\big|m_1, \ldots, m_r = 1, 2, \ldots\big\}, r = 1, 2, \ldots$. In the following we write $*$ for the $n \to \infty$ limit of the associative product $*_n$ defined in appendix C (where $x^{\mu_1\cdots\mu_r}$ has to be replaced by $t_{m_1\ldots m_r}$). Then (C.2) reads

$$(f * g)_{t_{m_1\ldots m_r}} = f_{t_{m_1\ldots m_r}} * g + f * g_{t_{m_1\ldots m_r}} + \sum_{k=1}^{r-1} f_{t_{m_1\ldots m_k}} * g_{t_{m_{k+1}\ldots m_r}} \tag{8.1}$$

and the first of these differentiation rules are

$$(f * g)_{t_m} = f_{t_m} * g + f * g_{t_m}$$
$$(f * g)_{t_{mn}} = f_{t_{mn}} * g + f * g_{t_{mn}} + f_{t_m} * g_{t_n}$$
$$(f * g)_{t_{mnr}} = f_{t_{mnr}} * g + f * g_{t_{mnr}} + f_{t_m} * g_{t_{nr}} + f_{t_{mn}} * g_{t_r}.$$

Applying them repeatedly, we find, e.g.,

$$\begin{aligned}(f * g * h)_{t_{mnr}} &= f_{t_{mnr}} * g * h + f * g_{t_{mnr}} * h + f * g * h_{t_{mnr}} \\ &\quad + f_{t_{mn}} * g_{t_r} * h + f_{t_{mn}} * g * h_{t_r} + f_{t_m} * g_{t_{nr}} * h + f_{t_m} * g * h_{t_{nr}} \\ &\quad + f * g_{t_{mn}} * h_{t_r} + f * g_{t_m} * h_{t_{nr}} + f_{t_m} * g_{t_n} * h_{t_r}. \end{aligned} \tag{8.2}$$

For $\xi_i = \sum_{m=1}^{\infty} t_m p_i^m$ with parameters p_i we obtain, for example,

$$\mathrm{e}^{\xi_1} * \mathrm{e}^{\xi_2} * \mathrm{e}^{\xi_3} = \mathrm{e}^{\xi_1+\xi_2+\xi_3+\xi_{12}+\xi_{13}+\xi_{23}+\xi_{123}} \tag{8.3}$$

where

$$\xi_{i_1\ldots i_r} := \sum_{m_1,\ldots,m_r=1}^{\infty} t_{m_1\ldots m_r} p_{i_1}^{m_1} \ldots p_{i_r}^{m_r}. \tag{8.4}$$

More generally,

$$\mathrm{e}^{\xi_1} * \cdots * \mathrm{e}^{\xi_N} = \exp\left(\sum_{r=1}^{N} \sum_{1\leqslant i_1<\cdots<i_r\leqslant N} \xi_{i_1\ldots i_r}\right) \tag{8.5}$$

which implies

$$\begin{aligned}(\mathrm{e}^{\xi_1} * \cdots * \mathrm{e}^{\xi_N})_{t_{m_1\ldots m_r}} &= \left(\sum_{1\leqslant i_1<\cdots<i_r\leqslant N} p_{i_1}^{m_1} \ldots p_{i_r}^{m_r}\right) \mathrm{e}^{\xi_1} * \cdots * \mathrm{e}^{\xi_N} \\ &= \Sigma_N\left(P_{m_1} \prec \cdots \prec P_{m_r}\right) \mathrm{e}^{\xi_1} * \cdots * \mathrm{e}^{\xi_N}\end{aligned} \tag{8.6}$$

using (3.8) in the last step.

8.2. *The XncKP hierarchy*

Comparing the generalized derivation rule (6.8) with (8.1) and recalling theorem 6.1, it is consistent to set $\partial_{t_{m_1\ldots m_r}} = \delta_{m_1\ldots m_r}$ on $\mathcal{R}$. Then (6.22), respectively (6.23), leads to

$$\begin{aligned}\partial_{t_{m_1\ldots m_r}} L^n &= -\left[L^{m_1,\ldots,m_r}{}_{<0}, L^n\right]_* + \sum_{k=1}^{r-1} \left(\partial_{t_{m_1\ldots m_k}} L^n\right) * L^{m_{k+1},\ldots,m_r}{}_{<0} \\ &= \left[L^{m_1,\ldots,m_r}{}_{\geqslant 0}, L^n\right]_* - \sum_{k=1}^{r-1} \left(\partial_{t_{m_1\ldots m_k}} L^n\right) * L^{m_{k+1},\ldots,m_r}{}_{\geqslant 0}\end{aligned} \tag{8.7}$$

where $L^{m_1,\ldots,m_r}$ is given by (6.15) in terms of L. For $n = 1$, these are the generalized Lax equations

$$\begin{aligned}L_{t_{m_1\ldots m_r}} &= \left[L^{m_1,\ldots,m_r}{}_{\geqslant 0}, L\right] - \sum_{k=1}^{r-1} L_{t_{m_1\ldots m_k}} * L^{m_{k+1},\ldots,m_r}{}_{\geqslant 0} \\ &= -\left[L^{m_1,\ldots,m_r}{}_{<0}, L\right]_* + \sum_{k=1}^{r-1} L_{t_{m_1\ldots m_k}} * L^{m_{k+1},\ldots,m_r}{}_{<0}\end{aligned} \tag{8.8}$$

which (as a consequence of theorem 6.1) define a hierarchy of commuting flows which we call the *XncKP hierarchy*. It is easy to see that they are the integrability conditions of the linear system

$$L * \psi = \lambda\psi \qquad \psi_{t_{m_1\ldots m_r}} = L^{m_1,\ldots,m_r}{}_{\geqslant 0} * \psi. \tag{8.9}$$

Taking the residue of (8.8), after an x-integration we find

$$\phi_{t_{m_1\ldots m_r}} = \mathrm{res}(L^{m_1,\ldots,m_r}) = \Phi\big(P_{m_1\ldots m_r}\big). \tag{8.10}$$

Introducing coefficients $\sigma_k^{(m_1,\ldots,m_r)}$ via

$$L^{m_1,\ldots,m_r}{}_{<0} = (-1)^r \sum_{k=1}^{\infty} \sigma_k^{(m_1,\ldots,m_r)} * L^{-k} \tag{8.11}$$

(8.10) takes the form

$$\phi_{t_{m_1\ldots m_r}} = (-1)^r \sigma_1^{(m_1,\ldots,m_r)}. \tag{8.12}$$

With the help of $L^{m_1,\ldots,m_{r+1}} = -L^{m_1,\ldots,m_r}{}_{<0} * L^{m_{r+1}}$ one obtains the iteration formula

$$\sigma_k^{(m_1,\ldots,m_{r+1})} = \sigma_{k+m_{r+1}}^{(m_1,\ldots,m_r)} - \sum_{l=1}^{m_{r+1}-1} \sigma_l^{(m_1,\ldots,m_r)} * \sigma_k^{(m_{r+1}-l)} \tag{8.13}$$

which corresponds to the identity (4.18) in $\mathcal{A}(P)$. The coefficients $\sigma_k^{(m)}$ already appeared in section 7.1 (see also (5.7) and (5.8) in [6]). An example from the set of equations (8.12) is

$$\begin{aligned}\phi_{t_{1,2,1}} &= -\sigma_1^{(1,2,1)} = -\sigma_2^{(1,2)} = -\sigma_4^{(1)} + \sigma_1^{(1)} * \sigma_2^{(1)} \\ &= \tfrac{1}{4}\phi_{t_4} - \tfrac{1}{3}\phi_{t_1t_3} - \tfrac{1}{8}\phi_{t_2t_2} + \tfrac{1}{4}\phi_{t_1t_1t_2} - \tfrac{1}{24}\phi_{t_1t_1t_1t_1} + \tfrac{1}{2}\phi_{t_1} * \big(\phi_{t_2} - \phi_{t_1t_1}\big).\end{aligned} \tag{8.14}$$

In a similar way, defining η-coefficients via

$$r\big(P_{m_r} \succ \cdots \succ P_{m_1}\big)_{<0} = \sum_{k=1}^{\infty} L^{-k} * \eta_k^{(m_1,\ldots,m_r)} \tag{8.15}$$

one obtains the expression

$$\eta_k^{(m_1,\ldots,m_{r+1})} = \eta_{k+m_{r+1}}^{(m_1,\ldots,m_r)} + \sum_{l=1}^{m_{r+1}-1} \eta_k^{(m_{r+1}-l)} * \eta_l^{(m_1,\ldots,m_r)} \tag{8.16}$$

which corresponds to the identity (4.15) in $\mathcal{A}(P)$. In fact, (8.13) and the last relation follow directly from the corresponding relations in $\mathcal{A}(P)$ by use of the following result.

Theorem 8.1.

$$\Phi\big(H_k^{(m_1,\ldots,m_r)}\big) = \eta_k^{(m_1,\ldots,m_r)} \qquad \Phi\big(C_k^{(m_1,\ldots,m_r)}\big) = \sigma_k^{(m_1,\ldots,m_r)}. \tag{8.17}$$

Proof. With the help of (6.40) and (6.3), we obtain

$$\begin{aligned}\Phi\big(H_k^{(m_1,\ldots,m_r)}\big) &= \Phi\big(H_{k-1} \succ P_{m_r} \succ \cdots \succ P_{m_1}\big) = \mathrm{res}\big(\ell(H_{k-1}) * r\big(P_{m_r} \succ \cdots \succ P_{m_1}\big)_{<0}\big) \\ &= \sum_{l=1}^{\infty} \mathrm{res}(\ell(P^{\succ k-1}) * L^{-l}) * \eta_l^{(m_1,\ldots,m_r)} \\ &= \sum_{l=1}^{\infty} \mathrm{res}(\ell(P^{\succ k-2})_{\geqslant 0} * L^{1-l}) * \eta_l^{(m_1,\ldots,m_r)} \\ &= \sum_{l=1}^{\infty} \mathrm{res}(\ell(P^{\succ k-2}) * (L^{1-l})_{<0}) * \eta_l^{(m_1,\ldots,m_r)}\end{aligned}$$

$$= \sum_{l=2}^{\infty} \mathrm{res}(\ell(P^{\succ k-2}) * L^{1-l}) * \eta_l^{(m_1,\dots,m_r)} = \cdots$$

$$= \sum_{l=k-1}^{\infty} \mathrm{res}(\ell(P) * L^{k-2-l}) * \eta_l^{(m_1,\dots,m_r)} = \sum_{l=k}^{\infty} \mathrm{res}(L^{k-1-l}) * \eta_l^{(m_1,\dots,m_r)}.$$

The second relation of the theorem is verified in a similar way. □

Explicit equations of the XncKP hierarchy are more generally obtained by application of Φ to identities in $\mathcal{A}(P)$ built from any subset of the elements $P_{m_1\dots m_k}$ and the products $\circ$ and $\hat{\times}$ (using (8.10), theorem 6.2 and proposition 6.5).

There is a redundancy in the parameters $t_{m_1\dots m_k}$. The remark at the end of section 7.2, which also applies to the more general $*$-product under consideration, shows that we may drop the symmetric part of t_{mn}. But now there are further identities in $\mathcal{A}(P)$ which lead to *linear* equations for ϕ and allow us to eliminate partial derivatives of ϕ with respect to certain combinations of the variables $t_{m_1\dots m_k}$ for fixed k. For example, with the help of (2.12), (2.7) and (7.15), we obtain

$$\begin{aligned} P_m \circ P_n \circ P_r = {} & P_{mnr} + P_{mrn} + P_{nrm} + P_{nmr} + P_{rmn} + P_{rnm} \\ & + P_m \circ P_{n+r} + P_n \circ P_{m+r} + P_r \circ P_{m+n} - 2P_{m+n+r} \end{aligned} \tag{8.18}$$

which is mapped by Φ to

$$\phi_{t_{mnr}} + \phi_{t_{mrn}} + \phi_{t_{nrm}} + \phi_{t_{nmr}} + \phi_{t_{rmn}} + \phi_{t_{rnm}} = \phi_{t_m t_n t_r} - \phi_{t_m t_{n+r}} - \phi_{t_n t_{m+r}} - \phi_{t_r t_{m+n}} + 2\phi_{t_{m+n+r}} \tag{8.19}$$

as a consequence of which the totally symmetric part of t_{mnr} turns out to be redundant. In particular, the last equation implies

$$\phi_{t_{mmm}} = \tfrac{1}{6}\phi_{t_m t_m t_m} - \tfrac{1}{2}\phi_{t_m t_{2m}} + \tfrac{1}{3}\phi_{t_{3m}}. \tag{8.20}$$

A similar calculation yields

$$P_{mnr} - P_{mrn} + P_{nrm} + P_{nmr} - P_{rmn} - P_{rnm} = 2(P_m \circ A_{nr} - A_{n,m+r} + A_{r,m+n}) \tag{8.21}$$

and anti-symmetrization with respect to m, n, r leads to

$$P_{mnr} - P_{mrn} + P_{nrm} - P_{nmr} + P_{rmn} - P_{rnm} = 2(P_m \circ A_{nr} + P_n \circ A_{rm} + P_r \circ A_{mn}) \tag{8.22}$$

so that, in particular, the totally antisymmetric part of t_{mnr} is redundant. Of course, (8.21) determines further redundancies. These are given by

$$P_{mmr} - P_{rmm} = P_m \circ A_{mr} - A_{m,m+r} + A_{r,2m} \qquad r \neq m \tag{8.23}$$

and additional relations with m, n, r pairwise different.

Let us look at some concrete examples. Application of Φ to the identity

$$P_{1,1,1} + P_{1,2} + P \hat{\times} P = P \prec P \prec P + P \prec P_2 + P \hat{\times} P = 0 \tag{8.24}$$

leads to the nonlinear XncKP equation

$$\phi_{t_{1,1,1}} + \phi_{t_{1,2}} + \phi_{t_1} * \phi_{t_1} = 0. \tag{8.25}$$

By use of the linear equation (8.20) this becomes

$$\tfrac{1}{3}\phi_{t_3} - \tfrac{1}{2}\phi_{t_1 t_2} + \tfrac{1}{6}\phi_{t_1 t_1 t_1} + \phi_{t_{1,2}} + \phi_{t_1} * \phi_{t_1} = 0 \tag{8.26}$$

which, with the help of the linear equation (7.17), is turned into an xncKP equation,

$$\phi_{\theta_{1,2}} - \tfrac{1}{6}\big(\phi_{t_3} - \phi_{t_1 t_1 t_1}\big) + \phi_{t_1} * \phi_{t_1} = 0. \tag{8.27}$$

Of course, this equation is obtained more directly from the identity

$$A_{1,2} - \tfrac{1}{6}(P_3 - P^{\circ 3}) + P \hat{\times} P = 0. \tag{8.28}$$

Furthermore, the identity

$$P_{1,2,1} = P \prec P_2 \prec P = P \prec P \succ P \prec P - P \prec P \prec P \prec P = -P \,\hat{\times}\, P^{\prec 2} - P^{\prec 4} \quad (8.29)$$

leads to the nonlinear XncKP equation

$$\phi_{t_{1,2,1}} = -\phi_{t_1} * \phi_{t_{1,1}} - \phi_{t_{1,1,1,1}} \quad (8.30)$$

where we should substitute the following expressions (obtained from (4.36), for example),

$$\phi_{t_{1,1}} = -\tfrac{1}{2}\phi_{t_2} + \tfrac{1}{2}\phi_{t_1 t_1} \quad (8.31)$$

$$\phi_{t_{1,1,1,1}} = -\tfrac{1}{4}\phi_{t_4} + \tfrac{1}{3}\phi_{t_1 t_3} + \tfrac{1}{8}\phi_{t_2 t_2} - \tfrac{1}{4}\phi_{t_1 t_1 t_2} + \tfrac{1}{24}\phi_{t_1 t_1 t_1 t_1} \quad (8.32)$$

which results in (8.14). Expressions for $\phi_{t_{1,1,2}}$ and $\phi_{t_{2,1,1}}$ are then obtained with the help of linear equations given above. We may take the view, however, that the dependence of ϕ on $t_{1,1,2}$, respectively $t_{2,1,1}$, is redundant (after selection of the variable $t_{1,2,1}$).

8.3. Reductions

Let us impose the constraint $(L^N)_{<0} = 0$ for some fixed $N \in \mathbb{N}$ which is known to reduce the KP hierarchy to the Nth Gelfand–Dickey hierarchy (see [3], for example). It immediately follows from (8.7) that all equations of the *XncKP* hierarchy preserve this constraint. Another immediate consequence is $(L^{kN})_{<0} = 0$ and thus $L_{t_{kN}} = 0$ for all $k \in \mathbb{N}$. Moreover, (6.15) shows that

$$L^{kN,m_2,\ldots,m_r} = 0 \qquad k \geqslant 1, \quad r \geqslant 2 \quad (8.33)$$

which, by use of (8.8), implies

$$L_{t_{kN,m_2\ldots m_r}} = 0 \qquad k \geqslant 1, \quad r \geqslant 2. \quad (8.34)$$

Furthermore,

$$\begin{aligned} L^{m_1,\ldots,m_r,kN}{}_{<0} &= \left(\left(L^{m_1,\ldots,m_{r-1}}{}_{<0} * L^{m_r}\right)_{<0} * L^{kN}\right)_{<0} \\ &= \left(L^{m_1,\ldots,m_{r-1}}{}_{<0} * L^{m_r} * L^{kN}\right)_{<0} - \left(\left(L^{m_1,\ldots,m_{r-1}}{}_{<0} * L^{m_r}\right)_{\geqslant 0} * (L^{kN})_{\geqslant 0}\right)_{<0} \\ &= -L^{m_1,\ldots,m_r+kN}{}_{<0} \qquad k \geqslant 1, \quad r \geqslant 1 \end{aligned} \quad (8.35)$$

by use of (6.11). With the help of (8.8), this leads to

$$L_{t_{m_1\ldots m_r,kN}} = -L_{t_{m_1\ldots m_r+kN}} \qquad k \geqslant 1, \quad r \geqslant 1. \quad (8.36)$$

Moreover, using (6.15) and (8.35), we obtain

$$L^{m_1,\ldots,m_{l-1},kN,m_{l+1},\ldots,m_r} = -L^{m_1,\ldots,m_{l-1}+kN,m_{l+1},\ldots,m_r} \qquad l = 2,\ldots,r-1, \quad r \geqslant 3, \quad k \geqslant 1 \quad (8.37)$$

and thus

$$L_{t_{m_1\ldots m_{l-1},kN,m_{l+1}\ldots m_r}} = -L_{t_{m_1\ldots,m_{l-1}+kN,m_{l+1}\ldots m_r}} \qquad l = 2,\ldots,r-1, \quad r \geqslant 3, \quad k \geqslant 1. \quad (8.38)$$

As an example, let us consider the KdV reduction $(L^2)_{<0} = 0$. In this case we have $\phi_{t_2} = 0$ and $\phi_{t_{1,2}} = -\phi_{t_3}$, so that (8.26) reduces to

$$\phi_{t_3} = \tfrac{1}{4}\phi_{t_1 t_1 t_1} + \tfrac{3}{2}\phi_{t_1} * \phi_{t_1} \quad (8.39)$$

which is the potential ncKdV equation. Furthermore, (8.14) reduces to

$$\phi_{t_{3,1}} = -\phi_{t_{1,2,1}} = \tfrac{1}{3}\phi_{t_1 t_3} + \tfrac{1}{24}\phi_{t_1 t_1 t_1 t_1} + \tfrac{1}{2}\phi_{t_1} * \phi_{t_1 t_1} \quad (8.40)$$

and (8.23) leads to the linear equation

$$\phi_{t_{1,3}} = -\phi_{t_{1,1,2}} = \phi_{\theta_{1,3}} + \tfrac{1}{2}\phi_{t_1 t_3} \quad (8.41)$$

with the help of which, and use of (8.39), the previous equation yields the xncKdV equation

$$\phi_{\theta_{1,3}} + \tfrac{1}{4}\left[\phi_{t_1}, \phi_{t_1 t_1}\right] = 0. \quad (8.42)$$

8.4. Generalized Sato–Wilson equations and Birkhoff factorization

The ncKP hierarchy can be formulated alternatively in terms of the Sato–Wilson equations

$$W_{t_m} = -(L^m)_{<0} * W \tag{8.43}$$

with (the dressing operator)

$$W = 1 + \sum_{n=1}^{\infty} w_n \partial^{-n}. \tag{8.44}$$

Since $t_1 = x$ and $L_{\geqslant 0} = \partial$, the case $m = 1$ leads to $W_x = -L_{<0} * W = \partial W - L * W$. Hence $L * W = W\partial$ or $L = W * \partial W^{-1}$, since W is invertible. The Sato–Wilson equations now take the form

$$W_{t_m} = -(W * \partial^m W^{-1})_{<0} * W. \tag{8.45}$$

These equations imply the Lax form (1.3) of the ncKP equations (see also [6]).

An obvious generalization of the above Sato–Wilson equations is given by

$$W_{t_{m_1\ldots m_r}} = -L^{m_1,\ldots,m_r}{}_{<0} * W. \tag{8.46}$$

They indeed imply the generalized Lax equations (8.8), as can be demonstrated by application of $\partial_{t_{m_1\ldots m_r}}$ to $L * W = W\partial$. From (8.46) we find

$$\begin{aligned} W_{t_{m_1\ldots m_r}} * W^{-1} &= -L^{m_1,\ldots,m_r}{}_{<0} = \left(L^{m_1,\ldots,m_{r-1}}{}_{<0} * L^{m_r}\right)_{<0} \\ &= \left(L^{m_1,\ldots,m_{r-1}}{}_{<0} * W * \partial^{m_r} W^{-1}\right)_{<0} \\ &= -\left(W_{t_{m_1\ldots m_{r-1}}} * \partial^{m_r} W^{-1}\right)_{<0} \end{aligned}$$

and thus the equivalent inductive form

$$W_{t_{m_1\ldots m_r}} = -\left(W_{t_{m_1\ldots m_{r-1}}} * \partial^{m_r} W^{-1}\right)_{<0} * W. \tag{8.47}$$

This can be rewritten as

$$W_{t_{m_1\ldots m_r}} = \left(W_{t_{m_1\ldots m_{r-1}}} * \partial^{m_r} W^{-1}\right)_{\geqslant 0} * W - W_{t_{m_1\ldots m_{r-1}}} \partial^{m_r} \tag{8.48}$$

and thus

$$(W * \mathrm{e}^{\hat{\xi}})_{t_{m_1\ldots m_r}} = L^{m_1,\ldots,m_r}{}_{\geqslant 0} * (W * \mathrm{e}^{\hat{\xi}}) \tag{8.49}$$

where $\hat{\xi} = \sum_{n\geqslant 1} t_n \partial^n$. Following [45] (see also [7, 46]), this leads to the Birkhoff factorization (generalized Riemann–Hilbert problem, see [25, 47] for example)

$$W(t) * \mathrm{e}^{\hat{\xi}(t)} = Y(t) * W(0) \tag{8.50}$$

with $Y = Y_{\geqslant 0}$. This is equivalent to

$$\mathrm{e}^{\hat{\xi}} W(0)^{-1} = W(t)^{-1} * Y(t) \tag{8.51}$$

since $W(t) \in G_-$ and $Y(t) \in G_+$, for the group $G = G_- G_+$ of ΨDOs.

Conversely, acting with $\partial_{t_{m_1\ldots m_r}}$ on (8.50), we get

$$\left(W(t)_{t_{m_1\ldots m_r}} + W(t)_{t_{m_1\ldots m_{r-1}}} \partial^{m_r}\right) * \mathrm{e}^{\hat{\xi}(t)} = Y(t)_{t_{m_1\ldots m_r}} * Y(t)^{-1} * W(t) * \mathrm{e}^{\hat{\xi}(t)} \tag{8.52}$$

and thus

$$W(t)_{t_{m_1\ldots m_r}} * W(t)^{-1} + W(t)_{t_{m_1\ldots m_{r-1}}} * \partial^{m_r} W(t)^{-1} = Y(t)_{t_{m_1\ldots m_r}} * Y(t)^{-1}. \tag{8.53}$$

Taking the $\mathcal{R}_{<0}$ part, noting that $\left(Y(t)_{t_{m_1\ldots m_r}} * Y(t)^{-1}\right)_{<0} = 0$ and $W(t)_{t_{m_1\ldots m_r}} * W(t)^{-1} = \left(W(t)_{t_{m_1\ldots m_r}} * W(t)^{-1}\right)_{<0}$, one recovers (8.47). Hence, the Birkhoff factorization (8.50) is equivalent to the XncKP hierarchy equations (8.8). Via (8.50) the space of solutions of the XncKP hierarchy is determined from the same initial data $W(0)$ as in the KP case [46].

9. Conclusions

Some crucial steps in this work are sketched in the following diagram.

$$\begin{tikzcd} \mathcal{A}(P) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & \mathcal{A}(P/Q) \arrow[d, "\Sigma_N"] \\ \text{XncKP} \arrow[r, "\text{trace method}"] & \text{algebraic sum identities} \end{tikzcd}$$

Our central object is the algebra $\mathcal{A}(P)$ generated by a single element P and supplied with certain associative products, which in particular give rise to a (mixable) shuffle product (and a Rota–Baxter algebra structure). The map Ψ embeds it into a corresponding algebra generated by two independent commuting elements P, Q. Identities in $\mathcal{A}(P)$ are then mapped by Ψ to identities in the latter algebra. These in turn are sent by Σ_N to algebraic sum identities in variables $p_n, q_n, n = 1, \ldots, N$. Since $N \in \mathbb{N}$ is arbitrary, this results in families of identities. Such identities were actually the starting point of this work. In the introduction we explained how algebraic identities of this kind emerge from the equations of the (nc)KP hierarchy via the 'trace method' [8]. It remained to find those families of identities in $\mathcal{A}(P)$ which correspond to KP equations. This is where the map Φ entered the stage. We found identities in $\mathcal{A}(P)$ which are mapped by Φ to KP equations and the whole hierarchy of KP equations expressed in the potential ϕ is recovered in this way (after setting the derivations δ_n equal to partial derivatives ∂_{t_n}).

Moreover, we found further families of identities and showed that these determine extensions of the ncKP hierarchy with deformed products. The xncKP hierarchy [5, 6] is rediscovered in this way. But we even discovered a new (XncKP) hierarchy which extends the xncKP hierarchy after deforming the product in a more general way.

The XncKP hierarchy contains linear equations and it seems that their existence is related to equivalence transformations of the $*$-product, which can be used to reduce the amount of deformation parameters (which correspond to evolution 'times' of the generalized hierarchy). This relation has not been sufficiently clarified in this work.

The fact that $(\mathcal{R}, ()_{\geqslant 0})$ (and also $(\mathcal{R}, ()_{<0})$) is a Rota–Baxter algebra (see appendix A) suggests generalizing the results of section 6 towards other Rota–Baxter algebras[29].

The correspondence between (X)ncKP equations and algebraic identities presented in this work sets up a bridge between different areas of mathematics. In view of the appearance of the KP hierarchy in many physical systems and in various mathematical problems, this should be an interesting new tool for further explorations. In particular, the KP hierarchy has deep relations with string theory (see [48–50], for example) and shows up in related models such as topological field theories [51–53] and matrix models [54, 55]. We should also mention its appearance in Seiberg–Witten theory [56] and a relation with random matrices [57]. We expect that deformations and extensions of the KP hierarchy will play a similar role and that interesting generalizations of these results can be achieved. Indeed, some motivation to study (Moyal-) deformations of the KP hierarchy originated from the following fact. In string theory, D-branes with a non-vanishing B-field are effectively described in a low energy limit by a Moyal-deformed Yang–Mills theory [58, 59]. Corresponding non-commutative instantons [60] are solutions of a Moyal-deformed self-dual Yang–Mills equation, from which Moyal-deformed soliton equations result by reductions, as in the classical case (see [61], for example).

[29] We may e.g. replace $\mathcal{R}$ by an algebra of Laurent series in an indeterminate λ, as in the AKNS hierarchy example [5, 7].

Such deformed soliton equations provide us with interesting examples of non-commutative field theories [60].

Within the framework of integrable systems our results suggest an apparently new method, namely to look for (series of) algebraic identities of a certain type in order to construct hierarchies of soliton equations. The XncKP hierarchy presented in this work was in fact discovered in this way.

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Appendix A. Rota–Baxter operators

We recall the Rota–Baxter relation of weight[30] q on a ring $\mathbb{A}$:

$$R(a)R(b) = R(R(a)b + aR(b)) - \mathrm{q}R(ab) \tag{A.1}$$

(see [14–17, 62]). A (not exhaustive) class of Rota–Baxter operators is obtained by the following construction [16, 62]. Given an endomorphism $\Lambda : \mathbb{A} \to \mathbb{A}$ of an algebra $\mathbb{A}$, i.e., $\Lambda(ab) = \Lambda(a)\Lambda(b)$ for all $a, b \in \mathbb{A}$,

$$R := \sum_{r\geqslant 1} \Lambda^r \tag{A.2}$$

(assuming convergence, or nilpotence for some power of Λ) defines a Rota–Baxter operator of weight -1. Also note that $\mathrm{id} + R$ is then a Rota–Baxter operator of weight 1. An important example, already presented by Baxter [14], is provided by the *standard Baxter algebra* [15, 17] of a set of generators $\{a, b, c, \ldots\}$ which are infinite sequences $a = (a_1, a_2, \ldots)$, with componentwise multiplication $ab = (a_1b_1, a_2b_2, \ldots)$ and the Rota–Baxter operator given by

$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots) \tag{A.3}$$

which is (3.10). This is of the above form with the shift operator $\Lambda(a_1, a_2, \ldots) := (0, a_1, a_2, \ldots)$. The standard Baxter algebra is naturally isomorphic to the free Baxter algebra on the same set of generators [15, 17]. The standard (or free) Baxter algebra with a single generator is isomorphic to the algebra of symmetric functions [17]. Another example is obtained by choosing $(\Lambda f)(x) := f(qx)$ on functions of a variable x, where q is a parameter. R is then the Jackson q-integral [17].

The following theorem [62] provides us with further examples and, in particular, shows that $(\mathcal{R}, *, ()_{\geqslant 0})$ and $(\mathcal{R}, *, ()_{<0})$ are Rota–Baxter algebras.

Theorem. *Let $(\mathbb{A}, +, \cdot)$ be a (with respect to the product $\cdot$ not necessarily commutative and not necessarily associative) ring. The following conditions are equivalent:*

(i) There is a Rota–Baxter operator R of weight 1 on $\mathbb{A}$ which is a group homomorphism of addition.

(ii) There are two subrings $\mathbb{A}_\pm$ of $\mathbb{A}$ and a subring $\mathbb{B}$ of $\mathbb{A}_+ \times \mathbb{A}_-$ (supplied with a ring structure in the obvious way by componentwise addition and multiplication) such that each element $a \in \mathbb{A}$ has a unique decomposition $a = a_+ + a_-$ with $(a_+, a_-) \in \mathbb{B}$.

[30] Via multiplication of the Rota–Baxter operator by q^{-1}, we can always achieve that a non-vanishing weight constant becomes equal to 1. In this sense, the weight constant is 'relatively unimportant' [16].

An idempotent *Rota–Baxter operator R of weight 1 is equivalent to a direct sum decomposition, i.e.,* $\mathbb{A} = \mathbb{A}_+ \oplus \mathbb{A}_-$.

Proof. Let us assume that (i) holds. Define $\mathbb{A}_+ := R(\mathbb{A})$ and $\mathbb{A}_- := (\mathrm{id} - R)(\mathbb{A})$. By assumption, $R(a) + R(b) = R(a + b)$. Furthermore, the Rota–Baxter relation $R(a)R(b) = R(R(a)b + aR(b) + ab)$ implies $R(a)R(b) \in R(\mathbb{A}) = \mathbb{A}_+$, so that $\mathbb{A}_+$ is a subring. Moreover, since $\mathrm{id} - R$ satisfies the same Rota–Baxter relation, $\mathbb{A}_-$ is also a subring. This supplies $\mathbb{A}_+ \times \mathbb{A}_-$ with a ring structure. Now

$$(R(a)R(b), (\mathrm{id} - R)(a)(\mathrm{id} - R)(b)) = (R(c), (\mathrm{id} - R)(c))$$

with $c := aR(b) + R(a)b - ab$ shows that there is a subring $\mathbb{B}$ of $\mathbb{A}_+ \times \mathbb{A}_-$ with the properties specified in (ii).

Conversely, if (ii) holds, $R(a) := a_+$ defines a homomorphism R with respect to the operation + and we have $a_- = (\mathrm{id} - R)(a)$. Now we compare the decomposition

$$aR(b) + R(a)b - ab = R(aR(b) + R(a)b - ab) + (\mathrm{id} - R)(aR(b) + R(a)b - ab)$$

with the identity

$$aR(b) + R(a)b - ab = R(a)R(b) - (\mathrm{id} - R)(a)(\mathrm{id} - R)(b)$$

where, as a consequence of the subring properties, the first term on the right-hand side lies in $\mathbb{A}_+$ and the second in $\mathbb{A}_-$. Since the decomposition of an element of $\mathbb{A}$ is unique, this implies

$$R(aR(b) + R(a)b - ab) = R(a)R(b)$$

which is the Rota–Baxter relation (of weight 1).

If R is idempotent, i.e., $R^2 = R$, one easily verifies that $\mathbb{A}_+ \cap \mathbb{A}_- = \{0\}$. Conversely, given $\mathbb{A} = \mathbb{A}_+ \oplus \mathbb{A}_-$, the projections onto the subrings define idempotent Rota–Baxter operators. □

The theorem also holds with 'ring' replaced by '$\mathbb{K}$-algebra' if R is $\mathbb{K}$-linear. If one of the conditions of the theorem is fulfilled, the *classical* **R**-*matrix* given by

$$\mathbf{R}(a) := a_+ - a_- \tag{A.4}$$

(which generalizes the Hilbert transform) satisfies

$$\mathbf{R}(a)\mathbf{R}(b) = \mathbf{R}(\mathbf{R}(a)b + a\mathbf{R}(b)) - ab \tag{A.5}$$

called the 'Poincaré–Bertrand formula' in [24] and the 'modified Rota–Baxter relation' in [22, 23]. Passing over to commutators, this yields the modified Yang–Baxter equation [24]. The product $\vartriangle$ used in section 6 can be expressed as follows [24],

$$a \vartriangle b = a_+b_+ - a_-b_- = \tfrac{1}{2}(\mathbf{R}(a)b + a\mathbf{R}(b)). \tag{A.6}$$

In terms of the Rota–Baxter operator given by $R(a) = a_+$, we have the following expression,

$$a \vartriangle b = R(a)b + aR(b) - ab. \tag{A.7}$$

Such a product, determined by a Rota–Baxter operator of weight 1, has been called 'double product' in [38] (see also [22, 23]). It is associative as a consequence of the Rota–Baxter relation.

We also refer to [28, 63–65] for explorations of Rota–Baxter algebras. In particular, according to [66] any Rota–Baxter algebra defines a dendriform trialgebra (see [67], for example)[31].

[31] Although the notation used in work on dendriform algebras looks similar to the notation used in section 2, one should note that the operations defining a dendriform algebra are *not* associative whereas our products are associative.

Appendix B. Some realizations of the algebra $\mathcal{A}$

In this appendix we briefly describe some realizations of the algebraic structure introduced in section 2, different from our main example of partial sum calculus in section 3.

Posets. A poset $\mathcal{P}$ is a set with a binary relation $i \leqslant j$ for $i, j \in \mathcal{P}$, such that

(i) for all i: $i \leqslant i$,
(ii) if $i \leqslant j$ and $j \leqslant i$, then $i = j$,
(iii) if $i \leqslant j$ and $j \leqslant k$, then $i \leqslant k$.

Let us write $i < j$ for $i \leqslant j$ and $i \neq j$. A finite non-empty subset $\{i_1, \ldots, i_n\}$ of $\mathcal{P}$ will be called a chain, if $i_1 < \cdots < i_n$. A chain I always has a smallest element $\min(I)$ and a greatest element $\max(I)$. The set $\mathcal{C}$ of chains of $\mathcal{P}$ is 'graded' with respect to the number of elements of the chains. Let $\mathcal{A}$ be the free vector space generated by $\mathcal{C}$ over $\mathbb{K}$ with basis vectors $\{e_I | I \in \mathcal{C}\}$. We define the algebraic structure as in the case of partial sums:

$$e_I \bullet e_J := \begin{cases} e_{I\cup J} & \text{if} \quad \max(I) = \min(J) \\ 0 & \text{otherwise} \end{cases} \tag{B.1}$$

$$e_I \prec e_J := \begin{cases} e_{I\cup J} & \text{if} \quad \max(I) < \min(J) \\ 0 & \text{otherwise} \end{cases} \tag{B.2}$$

and thus

$$e_I \succ e_J = \begin{cases} e_{I\cup J} & \text{if} \quad \max(I) \leqslant \min(J) \\ 0 & \text{otherwise.} \end{cases} \tag{B.3}$$

From these rules we find

$$e_I \circ e_J := \begin{cases} e_{I\cup J} & \text{if} \quad I \cup J \in \mathcal{C} \\ 0 & \text{otherwise.} \end{cases} \tag{B.4}$$

For a finite poset $\mathcal{P}$, we define a map $\Sigma : \mathcal{A} \to \mathbb{K}$ by $\Sigma(e_I) = 1$ for all $I \in \mathcal{C}$. Then, for $A_a = \sum_{i\in\mathcal{P}} a_{a,i} e_i$, $a = 1, \ldots, r$, we obtain

$$\Sigma(A_1 \circ \cdots \circ A_r) = \sum_{i_1,\ldots,i_r\in\mathcal{P}} c_{\{i_1,\ldots,i_r\}} a_{1,i_1} \cdots a_{r,i_r} \tag{B.5}$$

with

$$c_{\{i_1,\ldots,i_r\}} := \begin{cases} 1 & \text{if} \quad \{i_1, \ldots, i_r\} \in \mathcal{C} \\ 0 & \text{otherwise.} \end{cases}$$

A special example of a poset is given by a *rooted tree*, which possesses a distinguished element, the 'root', from which there is a unique path to any other element. The ordering of nodes along a path obviously defines poset relations $<$ and $\leqslant$. Then $R(A) := \sum_{n\in\mathcal{P}} \big(\sum_{k<n} a_k\big) e_n$, where $A = \sum_{n\in\mathcal{P}} a_n e_n$, defines a Rota–Baxter operator of weight -1 for the algebra $(\mathcal{A}^1, \bullet)$. Hence, with any rooted tree a Rota–Baxter algebra, and thus also a dendriform trialgebra [66], is associated.

The tensor product algebra of an associative algebra. Let $(\mathcal{A}^1, \bullet)$ be any associative algebra over $\mathbb{K}$, and $\mathcal{A}^r := \mathcal{A}^1 \otimes \cdots \otimes \mathcal{A}^1$ (r-fold tensor product over $\mathbb{K}$). Then $\mathcal{A} = \bigoplus_{r\geqslant 1} \mathcal{A}^r$ with the tensor product $\otimes$ is an associative algebra. The product $\bullet$ extends to an associative product in $\mathcal{A}$ by setting

$$(A_1 \otimes \cdots \otimes A_r) \bullet (B_1 \otimes \cdots \otimes B_s) := A_1 \otimes \cdots \otimes (A_r \bullet B_1) \otimes \cdots \otimes B_s \tag{B.6}$$

for all $A_1, \ldots, A_r, B_1, \ldots, B_s \in \mathcal{A}^1$. Let us now define a new associative product by

$$\alpha \succ \beta = \alpha \otimes \beta + \alpha \bullet \beta \qquad \forall \alpha, \beta \in \mathcal{A}. \tag{B.7}$$

Identifying $\otimes$ with $\prec$ in our general formalism, the main product $\circ$ becomes a 'mixable shuffle product', as considered in [27].

If $(\mathcal{A}^1, \bullet)$ is unital with unit E, we can define an operator $R : \mathcal{A} \to \mathcal{A}$ by $R(\alpha) := E \otimes \alpha$. This implies $R(\alpha) \bullet R(\beta) = R(\alpha \otimes \beta)$. The quasi-shuffle property leads to

$$R(\alpha) \circ R(\beta) = R(\alpha \circ R(\beta) + R(\alpha) \circ \beta + \alpha \circ \beta) \tag{B.8}$$

so that R is a Rota–Baxter operator of weight -1. The algebra $(\mathcal{A}, \circ, R)$ is the free Rota–Baxter algebra on $\mathcal{A}^1$ (of weight -1) [27]. The operator R satisfies

$$R(\alpha) \bullet R(\beta) + R^2(\alpha \bullet \beta) = R(\alpha \bullet R(\beta) + R(\alpha) \bullet \beta) \tag{B.9}$$

with respect to the $\bullet$-product. This is the condition in [68] for the map R to be *hereditary* and is called the *associative Nijenhuis relation* in [22, 23, 69, 70]. In fact, the following stronger identity holds,

$$R(\alpha \bullet \beta) = R(\alpha) \bullet \beta. \tag{B.10}$$

Appendix C. $*_n$ products

On the space of analytic functions (or formal power series) of the collection $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ of variables $x^{(1)} = \{x^\mu\}, x^{(2)} = \{x^{\mu\nu}\}, x^{(3)} = \{x^{\mu\nu\rho}\}, \ldots, x^{(n)} = \{x^{\mu_1\cdots\mu_n}\}$ (where the indices run over some discrete set) we introduce a product $*_n$ via[32]

$$(f *_n g)(x) := \exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} \sum_{k=0}^{r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_2^{\mu_{k+1}\ldots\mu_r}}\right) f(x_1) g(x_2)\bigg|_{x_1=x_2=0} \tag{C.1}$$

using the summation convention with respect to the indices μ_k. Obviously,

$$x^{\mu_1\ldots\mu_r} *_n x^{\nu_1\ldots\nu_s} = x^{\mu_1\ldots\mu_r} x^{\nu_1\ldots\nu_s} + x^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}$$

where the last term should be set to zero if the number of indices exceeds n. For $n = 1$ the product $*_n$ coincides with the ordinary one since

$$(f *_1 g)(x^{(1)}) = \exp\left(x^\mu\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu}\right)\right) f(x_1) g(x_2)\bigg|_{x_1=x_2=0} = f(x^{(1)}) g(x^{(1)}).$$

For $n = 2$ we find

$$\begin{aligned}(f *_2 g)(x^{(1)}, x^{(2)}) &= \exp\left(x^\mu\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu}\right)\right.\\ &\quad\left. + x^{\mu\nu}\left(\frac{\partial}{\partial x_1^{\mu\nu}} + \frac{\partial}{\partial x_2^{\mu\nu}} + \frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\right)\right) f\big(x_1^{(1)}, x_1^{(2)}\big) g\big(x_2^{(1)}, x_2^{(2)}\big)\bigg|_{x_1=x_2=0}\\ &= \exp\left(x^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\right) f\big(x^{(1)} + x_1^{(1)}, x^{(2)}\big) g\big(x^{(1)} + x_2^{(1)}, x^{(2)}\big)\bigg|_{x_1=x_2=0}\end{aligned}$$

which is the usual Moyal product (if the symmetric part of $x^{\mu\nu}$ vanishes).

Proposition. *The $*_n$-product is associative.*

[32] Here and in the following we should replace $\partial/\partial x^{\mu_1\cdots\mu_k}$ by 1 if $k = 0$ and $\partial/\partial x^{\mu_{k+1}\cdots\mu_r}$ by 1 if $k = r$.

Proof. According to the definition of the $*_n$-product, we have

$$(f *_n (g *_n h))(x) = \exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} \sum_{k=0}^{r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_2^{\mu_{k+1}\ldots\mu_r}}\right) \times \exp\left(\sum_{s=1}^{n} x_2^{\nu_1\ldots\nu_s} \sum_{l=0}^{s} \frac{\partial}{\partial x_3^{\nu_1\ldots\nu_l}} \frac{\partial}{\partial x_4^{\nu_{l+1}\ldots\nu_s}}\right) f(x_1)g(x_3)h(x_4)\bigg|_{\substack{x_1=x_2=0\\ x_3=x_4=0}} .$$

This depends on x_2 only through the second exponential. On functions which are *not* dependent on x_2, we find

$$\begin{aligned}
&\exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} \sum_{k=0}^{r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_2^{\mu_{k+1}\ldots\mu_r}}\right) \exp\left(\sum_{s=1}^{n} x_2^{\nu_1\ldots\nu_s} \sum_{l=0}^{s} \frac{\partial}{\partial x_3^{\nu_1\ldots\nu_l}} \frac{\partial}{\partial x_4^{\nu_{l+1}\ldots\nu_s}}\right)\\
&\quad = \exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} \sum_{k=0}^{r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \sum_{l=k}^{r} \frac{\partial}{\partial x_3^{\mu_{k+1}\ldots\mu_l}} \frac{\partial}{\partial x_4^{\mu_{l+1}\ldots\mu_r}}\right)\\
&\qquad \times \exp\left(\sum_{s=1}^{n} x_2^{\nu_1\ldots\nu_s} \sum_{l=0}^{s} \frac{\partial}{\partial x_3^{\nu_1\ldots\nu_l}} \frac{\partial}{\partial x_4^{\nu_{l+1}\ldots\nu_s}}\right)\\
&\quad = \exp\left(\sum_{s=1}^{n} x_2^{\nu_1\ldots\nu_s} \sum_{l=0}^{s} \frac{\partial}{\partial x_3^{\nu_1\ldots\nu_l}} \frac{\partial}{\partial x_4^{\nu_{l+1}\ldots\nu_s}}\right) \exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} S^{1,3,4}_{\mu_1\ldots\mu_r}\right)
\end{aligned}$$

where

$$\begin{aligned}
S^{1,3,4}_{\mu_1\ldots\mu_r} := {}& \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_r}} + \frac{\partial}{\partial x_3^{\mu_1\ldots\mu_r}} + \frac{\partial}{\partial x_4^{\mu_1\ldots\mu_r}} + \sum_{0<k<l<r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_3^{\mu_{k+1}\ldots\mu_l}} \frac{\partial}{\partial x_4^{\mu_{l+1}\ldots\mu_r}}\\
&+ \sum_{0<k<r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_3^{\mu_{k+1}\ldots\mu_r}} + \sum_{0<k<r} \frac{\partial}{\partial x_1^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_4^{\mu_{k+1}\ldots\mu_r}}\\
&+ \sum_{0<k<r} \frac{\partial}{\partial x_3^{\mu_1\ldots\mu_k}} \frac{\partial}{\partial x_4^{\mu_{k+1}\ldots\mu_r}}
\end{aligned}$$

is completely symmetric in the labels 1,3,4. As a consequence, we obtain

$$(f *_n (g *_n h))(x) = \exp\left(\sum_{r=1}^{n} x^{\mu_1\ldots\mu_r} S^{1,3,4}_{\mu_1\ldots\mu_r}\right) f(x_1)g(x_3)h(x_4)\bigg|_{x_1=x_3=x_4=0}$$

and a similar calculation yields the same expression for $((f *_n g) *_n h)(x)$. □

Partial differentiation with respect to $x^{\mu_1\ldots\mu_r}$ acts on a $*_n$-product as follows:

$$\frac{\partial}{\partial x^{\mu_1\ldots\mu_r}}(f *_n g) = \sum_{k=0}^{r} \frac{\partial f}{\partial x^{\mu_1\ldots\mu_k}} *_n \frac{\partial g}{\partial x^{\mu_{k+1}\ldots\mu_r}} \qquad r \leqslant n. \tag{C.2}$$

Suppose we impose an additional condition of the form

$$a_{\mu_1\ldots\mu_r} x^{\mu_1\ldots\mu_r} = 0 \tag{C.3}$$

with constants $a_{\mu_1\ldots\mu_r}$ on the deformation parameters. The multiplication rule (C.2) then leads to the further compatibility conditions

$$\begin{aligned}
0 &= x^{\nu} *_n \left(a_{\mu_1\ldots\mu_r} x^{\mu_1\ldots\mu_r}\right) = a_{\mu_1\ldots\mu_r} x^{\nu} *_n x^{\mu_1\ldots\mu_r}\\
&= x^{\nu}\left(a_{\mu_1\ldots\mu_r} x^{\mu_1\ldots\mu_r}\right) + a_{\mu_1\ldots\mu_r} x^{\nu\mu_1\ldots\mu_r} = a_{\mu_1\ldots\mu_r} x^{\nu\mu_1\ldots\mu_r}.
\end{aligned} \tag{C.4}$$

More generally, for all $p, q = 0, 1, 2 \ldots$ we find

$$a_{\mu_1\ldots\mu_r} x^{\nu_1\ldots\nu_p\mu_1\ldots\mu_r\rho_1\ldots\rho_q} = 0. \tag{C.5}$$

Appendix D. Left $\mathcal{A}(P)$-modules and Baker–Akhiezer functions

Let M be a left $\mathcal{A}$-module, so that $\alpha \prec m$ and $\alpha \bullet m$ are defined for all $\alpha \in \mathcal{A}$ and $m \in M$ with the following associativity relations,

$$(\alpha \prec \beta) \prec m = \alpha \prec (\beta \prec m) \qquad (\alpha \bullet \beta) \bullet m = \alpha \bullet (\beta \bullet m) \tag{D.1}$$

$$(\alpha \prec \beta) \bullet m = \alpha \prec (\beta \bullet m) \qquad (\alpha \bullet \beta) \prec m = \alpha \bullet (\beta \prec m). \tag{D.2}$$

We further assume that M is graded, i.e., $M = \bigoplus_{r\geqslant 0} M^r$ with $\mathcal{A}^r \prec M^s \subseteq M^{r+s}$ and $\mathcal{A}^r \bullet M^s \subseteq M^{r+s-1}$, and that M is completely determined by M^0 and $\prec$, so that $M^r \subseteq \mathcal{A}^r \prec M^0$. This reduces the left actions on M to the definitions of $A \prec \chi$ and $A \bullet \chi$ for $A \in \mathcal{A}^1$ and $\chi \in M^0$. Let $\alpha \succ m := \alpha \bullet m + \alpha \prec m$. Furthermore, for $\chi \in M^0$, we set

$$\alpha \circ \chi := \alpha \succ \chi \tag{D.3}$$

(which does *not* hold for general $m \in M$). The product $\circ$ then extends via the quasi-shuffle properties

$$(A \succ \alpha) \circ (B \succ m) = A \succ [\alpha \circ (B \succ m)] + B \succ [(A \succ \alpha) \circ m] - A \bullet B \succ \alpha \circ m \tag{D.4}$$

$$(A \prec \alpha) \circ (B \prec m) = A \prec [\alpha \circ (B \prec m)] + B \prec [(A \prec \alpha) \circ m] + A \bullet B \prec \alpha \circ m \tag{D.5}$$

$$(A \succ \alpha) \circ (B \prec m) = A \succ [\alpha \circ (B \prec m)] + B \prec [(A \succ \alpha) \circ m] \tag{D.6}$$

which are consistent with (D.3). By induction, one obtains

$$\alpha \circ (\beta \circ m) = (\alpha \circ \beta) \circ m \tag{D.7}$$

for $\alpha, \beta \in \mathcal{A}$ and $m \in M$. In fact, the proof is rather tedious and requires several generalizations of results obtained for the algebra $\mathcal{A}$.

In the following, we concentrate on a graded left $\mathcal{A}(P)$-module M. Let $M_{\mathcal{R}}$ be the left module of $\mathcal{R}$ containing the Baker–Akhiezer function of the ncKP hierarchy. We define a map $\tilde{\ell} : M \to M_{\mathcal{R}}$ by

$$\tilde{\ell}(\alpha \prec \chi) := -\ell(\alpha)_{<0} * \tilde{\ell}(\chi) \qquad \tilde{\ell}(\alpha \succ \chi) := \ell(\alpha)_{\geqslant 0} * \tilde{\ell}(\chi) \tag{D.8}$$

for all $\alpha \in \mathcal{A}$ and $\chi \in M^0$. This leads to

$$\tilde{\ell}(\alpha \bullet \chi) = \ell(\alpha) * \tilde{\ell}(\chi). \tag{D.9}$$

Furthermore, we define linear operators $\delta_{m_1\ldots m_r}$ on $M_{\mathcal{R}}$ by setting

$$\delta_{m_1\ldots m_r}\tilde{\ell}(m) := \tilde{\ell}\big(P_{m_1\ldots m_r} \circ m\big) \tag{D.10}$$

and requiring the generalized derivation rule

$$\delta_{m_1\ldots m_r}(X * \tilde{\ell}(\chi)) = \sum_{k=0}^{r} \big(\delta_{m_1\ldots m_k} X\big) * \delta_{m_{k+1}\ldots m_r}\tilde{\ell}(\chi) \tag{D.11}$$

for all $\chi \in \mathcal{R}$. Using (D.3) and (D.8), we obtain

$$\begin{aligned}\delta_{m_1\ldots m_r}\tilde{\ell}(\chi) &= \tilde{\ell}\big(P_{m_1\ldots m_r} \circ \chi\big) = \tilde{\ell}\big(P_{m_1\ldots m_r} \succ \chi\big) = \ell\big(P_{m_1\ldots m_r}\big)_{\geqslant 0} * \tilde{\ell}(\chi) \\ &= L^{m_1,\ldots m_r}{}_{\geqslant 0} * \tilde{\ell}(\chi).\end{aligned} \tag{D.12}$$

Let us call $\chi \in M^0$ a *Baker–Akhiezer element* if it satisfies

$$P \bullet \chi = \lambda \chi \tag{D.13}$$

with $\lambda \in \mathbb{K}$. Acting with $\tilde{\ell}$ on this equation leads to

$$L * \tilde{\ell}(\chi) = \lambda \tilde{\ell}(\chi). \tag{D.14}$$

Together with (D.12), this is equivalent to the linear system (8.9).

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